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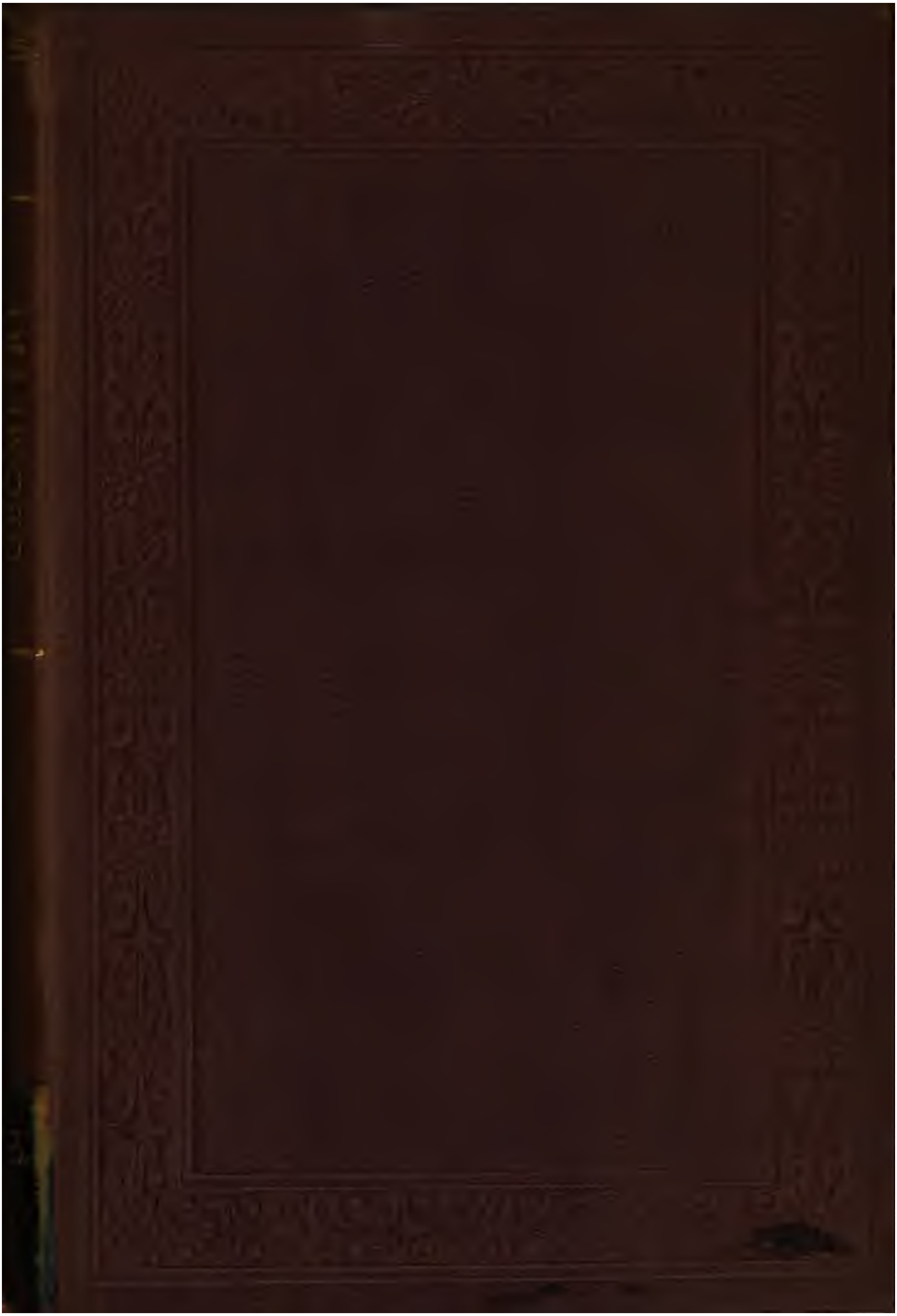

COORDINATE GEOMETRY

AND

CONIC SECTIONS.

AN ELEMENTARY MANUAL

OF

COORDINATE GEOMETRY

AND

CONIC SECTIONS.

BY THE

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BIBLIOTHECA
MAR 1878
BODLEIANA

LONDON:
C. F. HODGSON & SON, GOUGH SQUARE,
FLEET STREET.

1878.

183. g. 97.

LONDON:
PRINTED BY C. F. HODGSON AND SON,
GOUGH SQUARE, FLEET STREET.

PREFACE.

THIS book is chiefly an expansion of a syllabus on the subject drawn up for the use of the Cadets of the Royal Military Academy, by Professor CROFTON, F.R.S.; and on this syllabus the author based his lectures, the substance of which is here printed. When called away very unexpectedly to the Oxford Military College, it seemed to him undesirable that the labour expended, and the experience acquired, in organising this course should be lost; and consequently this little volume was, under pressure of many other engagements, drawn up so as to be ready for use at the commencement of the following term. Although the subject is treated specially with reference to the studies at the Royal Military Academy, still the book may be found more generally useful; and the author has accordingly resolved to publish it.

A very brief geometrical treatment of the properties of the Parabola, Ellipse, and Hyperbola has been added; so that a short course of Geometrical Conics may be read independently of, or before commencing, the Analytical treatment of the subject. This seems desirable, as, with a previous Geometrical knowledge of the curves, the pupil commences the Analysis with a distinct notion of what he is to learn; and Analytical methods, if studied without any reference to Pure Geometry, are apt to degenerate into mere manipulation of symbols without any real ideas attached to them.

An Appendix has been added containing a few of the more important elementary problems and theorems of Plane and Solid Geometry which are of frequent application; and sufficient hints have been given to enable the student who has finished Euclid to resolve them. This is independent of the rest of the work, and may be used first.

JAMES WHITE.

WOOLWICH;
January, 1878.

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COORDINATE GEOMETRY

AND

CONIC SECTIONS.

CHAPTER I.

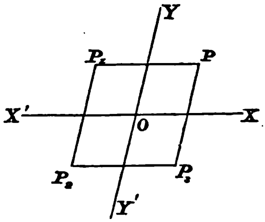
POSITION OF A POINT ON A PLANE.

1. ANY attentive learner of Euclid and elementary Algebra will have perceived a connection between them. Very often the same facts are proved in each. For example, most of the propositions of the Second Book of Euclid can be proved by Algebra, and in an easier and more expeditious manner than by Geometry. It is easier to see that $(a+b)^2 = a^2+b^2+2ab$, or that $(a-b)^2 = a^2+b^2-2ab$, than to prove the 4th or 7th proposition of the Second Book of Euclid. But while algebraical methods are usually more rapid, and more easily applied to given cases, than geometrical, the latter have this advantage, of conveying ideas more real—giving *pictures to the mind* of what the fact stated actually means. Any one can judge of this by comparing the geometrical and algebraical proofs of the above-mentioned propositions of the Second Book of Euclid; and for a student of Mathematics the knowledge and use of both methods is essential.

However, it would appear at first sight that the use of algebraical methods must be limited to those propositions which concern quantity only, and could not be applied to investigate the other properties of figures. This would be true were it not that means had been discovered of representing those other properties—position, direction, and form—by algebraical symbols. By this the methods of Algebra, with their greater generality, easiness of application, and facility of working, can be applied to purely geometrical questions; and many propositions which would be most difficult or impossible to pure Geometry can thus be easily solved.

This, which must be classed among the discoveries that have very greatly extended the boundaries of human knowledge, is sometimes called Cartesian Geometry, from its chief originator, the illustrious French philosopher, Des Cartes. It is also called Analytic Geometry, because it commences its investigations by supposing the thing done, or, in other words, reasons from the conditions given to some already known; and it is called Coordinate Geometry, from the means used to denote algebraically position and direction.

2. These means are as follows:—



In a plane, let a fixed point and two fixed straight lines intersecting through it be taken, and then the position of any other point in that plane can be expressed algebraically in reference to them. This fixed point is called the *origin*, and the two straight lines the *axes*. Let O be the origin, and OX and OY the axes: OX is called the axis of x, and OY the axis of y. The position of any point P in the plane, and within the angle XOY, can be denoted thus: Go a certain distance (a) along the axis of x, and then a certain distance (b) in the direction of the axis of y (*i. e.*, parallel to it), and that brings you to the point. It is the same as saying in the ordinary language of life, "Go so far to the right, and then so far to the left." The distance (a) in the direction of the axis of x is called the x of the point P, or the *abscissa* of the point; and the distance (b) in the direction of the axis of y is called the y of the point, or its *ordinate;* and the abscissa and ordinate are together called the *coordinates* of the point, whence the name Coordinate Geometry. The point P is called the point whose coordinates are a and b; or it may be denoted by the two equations $x=a$, $y=b$; it is often, for brevity's sake, termed the point a, b.

3. For any point within the angle $X'OY$ the abscissa is measured in an opposite direction to that for a point within XOY, while the direction of the ordinate is the same, therefore a negative sign is given to the x or abscissa. Thus the coordinates of the point P_1 (corresponding to P) are $x = -a$, $y = b$.

For any point within the angle $X'OY'$ the coordinates are both measured in the opposite direction from those taken as positive in the first case, and therefore both will be negative. Thus the coordinates of P_2 are $x = -a$, $y = -b$. And it will be seen that, for a point within the angle XOY', the x will be positive and the y negative: the coordinates of P_3 are $x = a$, $y = -b$.

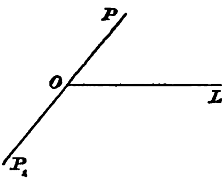
The reader will perceive that any real value, positive or negative, of x, and another real value, positive or negative, of y, being given, some point is by them denoted in the plane in which the origin and axes have been taken. It will also be easily seen that the equation $x = a$ must denote a straight line parallel to the axis of y, and at a distance from it (measured parallel to the axis of x) equal to a, because for any point in that line x must be equal to a. Also $x = -a$ represents a line also parallel to the axis of y, and at the same distance from it, but at the other side of the origin. Similarly, $y = b$, $y = -b$ represent lines parallel to the axis of x. The intersections of these lines give the four points P, P_1, P_2, P_3, whose coordinates are respectively $x = a, y = b$; $x = -a, y = b$; $x = -a, y = -b$; $x = a, y = -b$.

The axes may be taken intersecting at any angle. In general, it is most convenient that this should be a right angle, in which case it is said that *rectangular coordinates* are used. But sometimes *oblique-angled* coordinates may be used with advantage. Where it is not otherwise stated, rectangular coordinates are generally implied.

4. Another system of Analytic Geometry is used, which is called *polar coordinates.*

In this method the position of a point in a plane is fixed by the length of a line from it to a fixed point in the plane, called the *origin*, and by the angle which that line makes with a fixed line through the origin, called the *initial line.*

The line from the point to the origin is called the *radius vector*, and the angle which it makes with the initial line is called the *vectorial angle;* the radius vector being generally denoted by r, and the vectorial angle by θ. Thus, let O be the origin, OL the initial line, and P any point, its polar coordinates are $r = OP$, $\theta = LOP$. These being given, the point is evidently fixed.



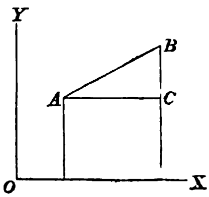
If the vectorial angle is greater than two right angles, the angle by which it exceeds 180° may be used if r is taken with a negative sign. Thus, let P_1 be a point taken in the production of OP through the origin. Let $OP_1 = OP = a$; then the polar coordinates of P_1 are $\theta = 180°+LOP$, and $r = a$, or $\theta = LOP$, $r = -a$: r is in this latter case taken negative, for it is measured in the opposite direction from the origin to that direction used in determining the vectorial angle θ.

5. Polar coordinates may be very simply connected with rectangular coordinates. Take the initial line as axis of x, and a line perpendicular to it through the origin as axis of y. Then it is evident that the ordinate of the point $y = r\sin\theta$, and the abscissa $x = r\cos\theta$, and $r = \sqrt{x^2+y^2}$. The relation may be thus expressed:

$$r^2 = x^2+y^2, \quad \theta = \tan^{-1}\frac{y}{x}.$$

By these equations the rectangular coordinates of a point may be easily turned into the polar coordinates, or *vice versâ.*

6. The distance between two points can be easily expressed in terms of their coordinates.



Let A and B be two points whose coördinates are respectively x_1, y_1 and x_2, y_2. Draw AC parallel to the axis of x, meeting y_2 in C; then, the axes being rectangular,

$$\begin{aligned} AB^2 &= AC^2+CB^2 \\ &= (x_2-x_1)^2+(y_2-y_1)^2. \end{aligned}$$

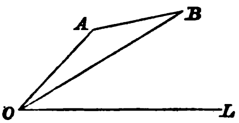
If the axes are oblique, let the angle between them be ω; then the angle ACB is $180^\circ - \omega$; therefore

$$AB^2 = AC^2 + CB^2 - 2AC \,.\, CB \cos ACB$$
$$= (x_2-x_1)^2 + (y_2-y_1)^2 + 2\,(x_2-x_1)\,(y_2-y_1) \cos \omega.$$

In polar coordinates, let OA, the radius vector of the point A, be r_1, and its vectorial angle θ_1; let OB, the radius vector of B, be r_2, and its vectorial angle θ_2; then

$$AB^2 = OA^2 + OB^2$$
$$-2OA \,.\, OB \cos AOB,$$

i.e., $= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$.



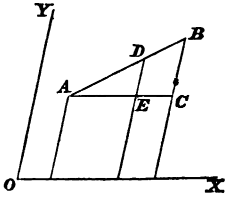
The distance of a point from the origin may be found by making the coordinates of one of the points in the above expressions both 0. With rectangular axes it is manifestly the square root of the sum of the squares of the coordinates of the point.

7. It is very useful to know the rule for finding the coordinates of a point which divides in a given ratio the line joining two points whose coordinates are given. In this case the reasoning and result are equally applicable to oblique-angled and rectangular axes. Let the coordinates of the points A and B be x_1, y_1 and x_2, y_2, as before; let the proportion in which the line joining them is divided be $m : n$, and the required coordinates of the point D be x, y; draw AC parallel to the axis of x, meeting y in E. From this,

$$AE : EC = AD : DC = m : n,$$

i.e., $x - x_1 : x_2 - x = m : n$;

therefore $\dfrac{x-x_1}{x_2-x} = \dfrac{m}{n}$. Solving for x, we find $x = \dfrac{mx_2 + nx_1}{m+n}$.

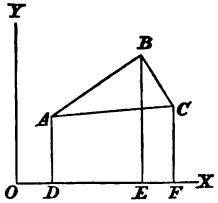


In like manner, $DE : BC = m : m+n = y - y_1 : y_2 - y_1$, or $m\,(y_2 - y_1) = (m+n)\,(y - y_1)$; from this, $y = \dfrac{my_2 + ny_1}{m+n}$.

If the point is external, the *plus* sign becomes a *minus* in these expressions.

If the point divides the line equally, $m=n$, and $x=\frac{1}{2}(x_2+x_1)$, and $y=\frac{1}{2}(y_2+y_1)$. That is, the coordinates of the middle point of a line are each half the sum of the respective coordinates of its extremities.

8. Another problem, easy and useful, is to express the area of a triangle in terms of the coordinates of the angular points. Let the coordinates of A, B, C be respectively x_1, y_1; x_2, y_2; x_3, y_3: and (the axes being rectangular) draw AD, BE, and CF perpendicular to the axis of x. The area of the triangle ABC = the trapezium $DABE$ + the trapezium $EBCF$ − the trapezium $DACF$. Now the area of the trapezium $DABE$ $=\frac{1}{2}(AD+BE)\,DE$. This is manifest, as it may be divided into two triangles ADE and ABE, of which the bases are AD and BE respectively, and the common perpendicular is DE.



Also the trapezium $EBCF=\frac{1}{2}(BE+CF)\,EF$;

and the trapezium $DACF=\frac{1}{2}(AD+CF)\,DF$.

Now $AD=y_1$, $BE=y_2$, $CF=y_3$, and $DE=x_2-x_1$, $EF=x_3-x_2$, $DF=x_3-x_1$. Consequently the area of the

trapezium $DABE=\frac{1}{2}(y_1+y_2)(x_2-x_1)$;

and of the trapezium $EBCF=\frac{1}{2}(y_2+y_3)(x_3-x_2)$;

and of the trapezium $DACF=\frac{1}{2}(y_1+y_3)(x_3-x_1)$.

Therefore the area of the triangle ABC

$$=\tfrac{1}{2}\{(y_1+y_2)(x_2-x_1)+(y_2+y_3)(x_3-x_2)-(y_1+y_3)(x_3-x_1)\};$$

or (changing the last sign and the order in the last bracket for symmetry)

$$=\tfrac{1}{2}\{(y_1+y_2)(x_2-x_1)+(y_2+y_3)(x_3-x_2)+(y_1+y_3)(x_1-x_3)\}.$$

This can be easily reduced to the form

$$\tfrac{1}{2}(x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_1y_3-x_3y_1).$$

This is the simplest expression for the area of a triangle in terms of the coordinates of its angular points.

If the axes be oblique and inclined at angle ω, then the expression must be multiplied by $\sin\omega$. This is evident, for the area of each trapezium must be multiplied by $\sin\omega$. (The student can draw the figure.) The perpendicular of the two triangles into which $DABE$, for instance, can be divided, would be $DE \sin\omega$; consequently its area would be

$$\tfrac{1}{2}(y_1+y_2)(x_1-x_2)\sin\omega.$$

If the expression within the bracket for the area of a triangle be put equal to 0, it is the condition that the three points A, B, C should be *in directum;* for the area of the triangle vanishes.

EXERCISES.

1. Find the polar coordinates of the points whose rectangular coordinates are: (1) $x=1$, $y=1$; (2) $x=4$, $y=3$; (3) $x=1$, $y=-1$.

2. Find the rectangular coordinates of the points whose polar coordinates are: (1) $r=4$, $\theta=\frac{1}{3}\pi$; (2) $r=-4$, $\theta=\frac{1}{3}\pi$; (3) $r=-4$, $\theta=-\frac{1}{4}\pi$.

3. Find the sides of the triangle the coordinates of whose angular points are (4, 3), (−3, 1), (1, −2).
[*Ans.* $\sqrt{51}$, 5, 5.

4. Also of the triangle whose angular points are (2, 3), (4, −5), (−3, −6). [*Ans.* $\sqrt{68}$, $\sqrt{50}$, $\sqrt{106}$.

5. Find the lengths of the sides of the triangle the coordinates of whose angular points are the same as in the last example, but the axes being inclined at an angle of 60°.
[*Ans.* $\sqrt{52}$, $\sqrt{57}$, $\sqrt{151}$.

6. Find the area of the triangle given in Example 3.
[*Ans.* $\frac{29}{2}$.

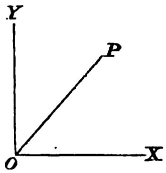
7. The straight line joining the points (2, 3), (4, −5) is trisected: find the coordinates of the two points.
[*Ans.* $(\frac{8}{3}, \frac{1}{3})$, $(\frac{10}{3}, -\frac{7}{3})$.

CHAPTER II.

EQUATION TO THE STRAIGHT LINE.

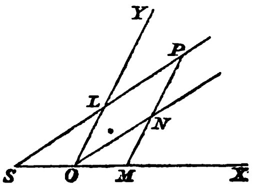
9. HAVING seen how a point and its relations to other points can be represented by algebraical symbols, the student must proceed to learn how straight lines and their relations can be similarly represented. In one case this has been anticipated; $x = c$ (where c is any constant) having been shown to be a line parallel to the axis of y, and distant from it by a length measured parallel to the axis of x equal to c. In like manner, $y = k$ represents a line similarly placed with regard to the axis of x. In fact, the coordinates of a point $x = a$, $y = b$ are the equations to two straight lines parallel to the axes, and the point is their intersection, its coordinates satisfying both equations.

10. The next simplest form of equation to a straight line is when the line passes through the origin. In that case $y = mx$ represents the line, for it is true for every point on the line, and for no other points in the plane; the coordinates of any point P in the line having manifestly the same ratio to each other.



Also, it is evident that $m = \frac{y}{x}$, which in rectangular axes $= \tan XOP$, *i.e.* $=$ tangent of the angle which the line makes with the axis of x.

When a line does not pass through the origin, its equation can be found thus:—Draw through the origin a line parallel to the given line, and let its equation be $y = mx$. Take any point P on the given line, draw its ordinate PM (which is the y of the point, and OM is the x of the point). $PM = MN + PN$. But MN, being on the line



through the origin, $= mOM = mx$. Therefore $y = mx + PN$, and $PN = OL$, which is a fixed length, being the intercept which the given line makes on the axis of y. Calling OL c, the required equation is $y = mx + c$, where, with rectangular axes, $m = \tan MON$, $= \tan LSO$, $=$ tangent of the angle which the line makes with the axis of x, &c. $= OL$, $=$ the intercept which the line makes on the axis of y.

Any equation of the first degree can be turned into this form. $Ax + By + C = 0$ can be written thus: $y = -\frac{A}{B}x - \frac{C}{B}$. Consequently it represents a straight line making, with the axis of x, an angle whose tangent is $-\frac{A}{B}$, and an intercept on the axis of y whose length from the origin is $-\frac{C}{B}$. It will be observed that the negative sign to the tangent of the angle denotes that the line is inclined to the *left* instead of to the *right;* and the negative sign of the indeterminate coefficient shows that the intercept on the axis of y is measured *below* the origin at a distance $= \frac{C}{B}$.

The student should exercise himself in observing the various positions of the line as the signs are varied in the general equation $Ax + By + C = 0$.

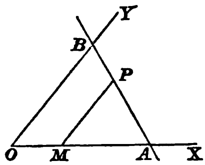
From this it will be seen that an equation of the first degree, whose coefficients are real, always represents *some* straight line. It will be afterwards shown that it can only represent a straight line.

When the axes are inclined at an oblique angle $= \omega$, then $m = \frac{y}{x} = \frac{\sin \alpha}{\sin (\omega - \alpha)}$, where α is the angle the line makes with the axis of x.

11. Some forms of the equation to the straight line are important to be remembered. For example:—

The equation to a straight line in terms of the intercepts it makes with the axes.

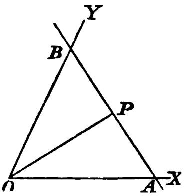
Let AB be the straight line, making OA and OB the intercepts on the axes of x and y respectively; take any point in it P, and draw the ordinate PM; then $PM=y$, $OM=x$, and let the intercepts OA and OB be called a and b. From similar triangles it follows that $\frac{PM}{OB}=\frac{AM}{AO}$, that is, $\frac{y}{b}=\frac{a-x}{a}$; therefore $\frac{x}{a}+\frac{y}{b}=1$, which is the required form of the equation.



The general equation $Ax+By+C=0$ can be brought into this form by bringing C to the right-hand side of the equation, and then dividing every coefficient by C. Then it becomes $-\frac{A}{C}x-\frac{B}{C}y=1$, which shows that the line makes intercepts in the negative sides of the axes equal to $\frac{A}{C}$ and $\frac{B}{C}$ respectively. Comparing it with the equation $y=mx+c$, the same results as those noted in the previous article as to the position and direction of the line are obtained.

A very easy method of drawing a line whose equation is given can be derived from this. Make $y=0$, and then the intercept on the axis of x is given; make $x=0$, and the value of y is the intercept on that axis, and the straight line joining these two points is the required one.

12. From this form of the equation to a straight line another very useful form can be derived, in terms of the perpendicular from the origin on the line and the angles which it (the perpendicular) makes with the axes. Let the length of this perpendicular $OP=p$, and the angle AOP which it makes with the axis of x be α, and the angle BOP which it makes with the axis of y be β. The equation to the straight line AB is, as proved above, $\frac{x}{a}+\frac{y}{b}=1$.



Multiply all the terms of this equation by p, which leaves it unchanged in signification, and it becomes $\frac{p}{a}x+\frac{p}{b}y=p$. But $\frac{p}{a}=\frac{OP}{OA}=\cos AOP=\cos\alpha$, similarly $\frac{p}{b}=\cos\beta$; therefore the equation becomes $x\cos\alpha+y\cos\beta=p$.

The two preceding articles apply to oblique-angled coordinates as well as rectangular. But when the axes are rectangular in the latter form of the equation, $\beta=90°-\alpha$. Consequently, in rectangular coordinates, the equation to a straight line, in terms of the perpendicular and the angle it makes with the axis of x, is $x\cos\alpha+y\sin\alpha=p$.

13. It is very useful and important to be able to turn into this form the general equation $Ax+By+C=0$. The rule for doing so may be proved thus:—The equation $\frac{x}{a}+\frac{y}{b}=1$ may be written $bx+ay=ab$, which is of the form $-Ax-By=C$. If they are identical and represent the same straight line, then the coefficients are proportional; therefore $-A:-B=b:a$, therefore

$$\frac{b}{\sqrt{a^2+b^2}}=\frac{-A}{\sqrt{A^2+B^2}},\quad \frac{a}{\sqrt{a^2+b^2}}=\frac{-B}{\sqrt{A^2+B^2}},$$

and
$$\frac{ab}{\sqrt{a^2+b^2}}=\frac{C}{\sqrt{A^2+B^2}}.$$

Referring to the last figure, (the axes being supposed rectangular, and AOB a right angle,) it is evident that, since $OL=a$, and $OM=b$, LM must be equal to $\sqrt{a^2+b^2}$ (the angle at O being right). Consequently,

$$\frac{b}{\sqrt{a^2+b^2}}=\frac{OM}{LM}=\sin BAO=\cos\alpha,$$

and
$$\frac{a}{\sqrt{a^2+b^2}}=\frac{OL}{LM}=\cos MLO=\sin\alpha;$$

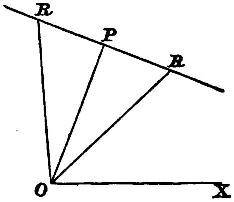
and since the area of the triangle $LOM=\frac{1}{2}OL\times OM=\frac{1}{2}ab$, and is also equal to $\frac{1}{2}p\times LM=\frac{1}{2}p\,.\sqrt{a^2+b^2}$, it follows that $\frac{ab}{\sqrt{a^2+b^2}}=p$. Therefore

$$\frac{A}{\sqrt{A^2+B^2}}=\cos a,\quad \frac{B}{\sqrt{A^2+B^2}}=\sin a,\quad \text{and}\quad \frac{-C}{\sqrt{A^2+B^2}}=p.$$

The general equation $Ax+By+C=0$ is turned into the required form by the following rule:—Change all the signs, and divide the coefficients by $\sqrt{A^2+B^2}$. It will be observed that $\left(\frac{A}{\sqrt{A^2+B^2}}\right)^2+\left(\frac{B}{\sqrt{A^2+B^2}}\right)^2=1$, so fulfilling the condition for representing $\cos a$ and $\sin a$.

POLAR EQUATION OF A STRAIGHT LINE.

14. Let O be the origin, OX the initial line, PR any given line whose equation it is proposed to find, and R any point on that line. Let OR, the radius vector, be called r; and the vectorial angle $XOR=\theta$; the perpendicular from the origin on the line OP being called p, and the angle XOP it makes with the initial line a. A relation can be easily established between the first two of three quantities, which are variable, and the two latter, which are for any given line fixed.



$$OR\cos ROP=OP;\quad\text{that is,}\quad r\cos(\theta-a)=p,$$

which is the polar equation of a straight line.

It will be observed that, when $\theta=90^\circ+a$, r becomes infinite. If the initial line coincides with the perpendicular from the origin, the equation becomes $r\cos\theta=p$.

This equation can be also obtained from $x\cos a+y\sin a=p$, by putting for x its equivalent $r\cos\theta$, and for y, $r\sin\theta$; then it becomes $r(\cos\theta\cos a+\sin\theta\sin a)=p$, which is equivalent to $r\cos(\theta-a)=p$.

15. The polar equation can also be derived from the general equation, with rectangular axes, $Ax+By+C=0$, by putting for x and y their values $r\cos\theta$ and $r\sin\theta$, by which it becomes $Ar\cos\theta+Br\sin\theta=-C$.

This can be brought back to the form $r\cos(\theta-\alpha)=p$ by dividing it by $\sqrt{A^2+B^2}$, when it is written thus: $r(A\cos\theta+B\sin\theta)=-C$; for it becomes

$$r\left(\frac{A}{\sqrt{A^2+B^2}}\cos\theta+\frac{B}{\sqrt{A^2+B^2}}\sin\theta\right)=\frac{-C}{\sqrt{A^2+B^2}};$$

therefore this is equivalent to

$$r(\cos\alpha\cos\theta+\sin\alpha\sin\theta)=p=r\cos(\theta-\alpha).$$

16. The following forms of the equation of the straight line have been given. The student should make himself familiar with them and their proofs and relations to each other:—

(1) $x=\text{constant}$, $y=\text{constant}$.
(2) $y=mx$. (3) $y=mx+c$.
(4) $Ax+By+C=0$. (5) $\frac{x}{a}+\frac{y}{b}=1$.
(6) $x\cos\alpha+y\cos\beta=p$. (7) $x\cos\alpha+y\sin\alpha=p$.
(8) $r\cos(\theta-\alpha)=p$. (9) $r\cos\theta=p$.
(10) $r(A\cos\theta+B\sin\theta)=-C$.

EXERCISES.

1. Draw the straight lines represented by the equations $y=3x+1$, $5y-4x=0$, $3x-2y+6=0$, $\frac{1}{3}x+\frac{1}{4}y=2$, $2x+y=0$, $\theta=\frac{1}{3}\pi$, $\theta=1$, $\tan(\theta-45^\circ)=1$.

2. Find the angles made with the axis of x, and the intercepts on both axes made by the straight lines whose equations are $2x-7y=2$, $3y-4x=6$, $4y-5x=20$.

CHAPTER III.

PROBLEMS ON THE STRAIGHT LINE.

17. *To find the coordinates of the point of intersection of two straight lines whose equations are given.*

Solve the two equations as two simple equations with two unknown quantities; the resulting values of x and y are the coordinates of the point required. For, as these coordinates satisfy each of the equations, the point they represent must be on both the lines, and must therefore be their point of intersection.

18. *To find the equation of a straight line parallel to another whose equation is given.*

Put the given equation into the form $y = mx + c$: since m is the tangent of the angle the line makes with the axis of x, the parallel line making the same angle with the axis of x must have for its equation $y = mx + c_1$. The equations of parallel straight lines therefore differ only in their indeterminate coefficient. This applies also to oblique-angled axes, as $m = \dfrac{\sin \alpha}{\sin(\omega - \alpha)}$.

19. *The equation of a straight line fulfilling two conditions can always be found.*

For, as the equation can always be put in the form $y = mx + c$, or in the form $x \cos \alpha + y \sin \alpha = p$, by treating the coefficients as unknown quantities, two are given, to find m and c, or α and p.

For example, to find the equation of a straight line parallel to another straight line and passing through a given point: In the first form of the equation m is given, and putting in the coordinates x', y' of the given point, the value of C is found to be $y' - mx'$. In the second form α is given, and p is found at once to be $x' \cos \alpha + y' \sin \alpha$.

20. *To find the equation of a straight line joining two points whose coordinates (x', y'), (x'', y'') are given.*

Here at once two equations are given to find the m and c of the required equation. The work is as follows:

$$\begin{array}{ll} & y' = mx' + c \\ \text{subtracting} & \underline{y'' = mx'' + c} \\ & y' - y'' = m(x' - x'') \end{array}$$

the value of m is found to be $\dfrac{y' - y''}{x' - x''}$.

Putting in this value of m in the first equation, we have

$$y' = \frac{y' - y''}{x' - x''} x' + c.$$

Therefore $$c = \frac{x'y'' - x''y'}{x - x''}.$$

The required equation is then found to be

$$y = \frac{y' - y''}{x' - x''} x + \frac{x'y'' - x''y'}{x' - x''}.$$

This can be written in the form

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'},$$

which is easily remembered and applied.

The same may be said for the form, which can also be given, $(y' - y'')x - (x' - x'')y + x''y' - x'y'' = 0$.

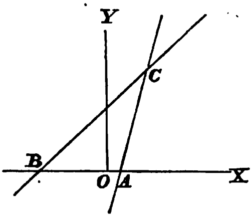
The following form is useful, as showing immediately the truth of the equation and some of its applications:

$$(x'' - x')(y - y') = (x - x')(y'' - y').$$

On inspection, it is evident that it must represent a straight line, and is satisfied by $x = x'$, $y = y'$, or $x = x''$, $y = y''$, and consequently both the points (x', y'), (x'', y'') are on the line represented by the equation. The equation of a line joining any point x', y' with the origin is evidently (making $x'' = 0$, $y'' = 0$) $y'x = x'y$.

The form $\dfrac{y - y'}{x - x'} = \dfrac{y'' - y}{x'' - x}$ may be very quickly found thus:—By subtraction as above, $m = \frac{y''-y'}{x''-x'}$. Now, if x, y be any other point on the line, it follows that $m = \frac{y-y'}{x-x'}$. Consequently, these two fractions are equal to each other.

21. *To find the angle between two straight lines whose equations are given.*



Let AC and BC be the lines. The angle between them $ACB =$ the angle CAX $-$ the angle CBX.

Therefore $\tan ACB$

$$= \frac{\tan CAX - \tan CBX}{1 + \tan CAX \times \tan CBX}.$$

Let the equation of AC be $y = mx + c$, and the equation of BC be $y = m'x + c'$; then $\tan CAX = m$, and $\tan CBX = m'$.

Consequently, $\tan ACB = \frac{m-m'}{1+m \cdot m'}$.

From this follows immediately the condition that two lines should be parallel; for $\tan ACB = 0$, therefore $m - m' = 0$, that is, $m = m'$.

22. *To find the condition of a straight line being perpendicular to another.*

In this case, the tangent of the angle between the two lines is infinite; therefore, in $\frac{m-m'}{1+m \cdot m'}$, $1 + m \cdot m' = 0$; from which $m = -\frac{1}{m'}$.

If the equation of one line be given in the form

$$x \cos a + y \sin a = p,$$

the condition manifestly is

$$x \sin a + y \cos a = p'.$$

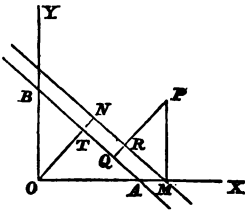
23. *To find the equation of a straight line which passes through a given point, and makes a given angle with another given straight line,* $y = mx + c$.

Since the line passes through a given point (x', y'), it follows (Art. 20) that $\frac{y-y'}{x-x'} = m'$. Since m' is the tangent of the angle that the required line makes with the axis of x, that angle must be equal to the given angle $(\beta) \pm$ the angle made by the given line with the axis of x, *i. e.*, the angle $\tan^{-1} m \; (= \alpha)$, for two lines can be drawn fulfilling the conditions.

$$\text{Therefore} \quad m' = \tan(\beta \pm \alpha) = \frac{\tan\beta \pm m}{1 \mp \tan\beta \, . \, m}.$$

$$\text{The equation therefore is} \quad \frac{y-y'}{x-x'} = \frac{\tan\beta \pm m}{1 \mp \tan\beta \, . \, m}.$$

24. *To find the length of a perpendicular from a given point on a given straight line.*



Let AB be the given line, and P the given point whose coordinates are x', y'. Draw the ordinate of the point $PM = y'$, and $OM = x'$. From M draw a line MN parallel to the given line; let fall PQ, the perpendicular whose length is required; let it meet MN in R. Draw the perpendicular OT from the origin on the given line, and let it meet MN in N; and OT will $= p$, and $MON = \alpha$, when the equation of the given line is in the form $x \cos\alpha + y \sin\alpha = p$. From the figure,

$$PQ = PR + ON - OT = PM \sin PMN + OM \cos\alpha - p.$$

But $PMN = 90° - OAB = \alpha$. Therefore the length of the perpendicular $PQ = y' \sin\alpha + x' \cos\alpha - p$; that is, the equation of the given line in the form $x' \cos\alpha + y' \sin\alpha - p$, with the coordinates of the given point put in respectively for x and y. This is an important result.

If the equation of the given straight line be in the form $Ax + By + C = 0$, it must be turned into the form

$x \cos \alpha + y \sin \alpha - p = 0$ (Art. 13), by changing all signs, so as to make the last term negative, and dividing the coefficients by $\sqrt{(A^2+B^2)}$, and then putting in the coordinates of the given point. Thus,

$$-\frac{A}{\sqrt{A^2+B^2}}x' - \frac{B}{\sqrt{A^2+B^2}}y' - \frac{C}{\sqrt{A^2+B^2}}$$

will be the length of the perpendicular from the point (x', y') on the line $Ax+By+C=0$. If the equation of the straight line is in the form $y = mx+c$, the length of the perpendicular on it from x', y' is $\frac{y'-mx'-c}{\sqrt{1+m^2}}$.

The length of the perpendicular from the origin, as in that case $x'=0$ and $y'=0$, is $\frac{-C}{\sqrt{A^2+B^2}}$.

25. *To find the equation of a straight line which passes through the intersection of two given straight lines.*

This may be done very shortly thus:—If $Ax+By+C=0$ be one of the given straight lines, and $A'x+B'y+C'=0$ be the other, then

$$Ax+By+C + k(A'x+B'y+C') = 0,$$

where k is any constant, must represent a straight line passing through their intersection. For it is the equation of a straight line, as it can be written in the form

$$(A+kA')x+(B+kB')y+C+kC' = 0;$$

and the values of x and y which satisfy the two given equations, making them each $=0$, must satisfy the third, making $0+k0=0$. Therefore the coordinates of the point of intersection of the two lines $Ax+By+C=0$ and $A'x+B'y+C'=0$ must satisfy

$$(Ax+By+C) + k(A'x+B'y+C') = 0.$$

An important general principle can be established immediately from the foregoing.

Three (or more) straight lines intersect in the same point when the sum of their equations, multiplied respectively by any constants, is equal to nothing. Upon this the system of abridged notation is founded. If L and L' are the equations to two straight lines, $L \pm kL' = 0$ must represent all straight lines through their intersection.

26. *To find the equations of the bisectors of the angles between two given straight lines.*

Let the equations be put in the form

$$x \cos \alpha + y \sin \alpha - p = 0, \quad x \cos \alpha' + y \sin \alpha' - p' = 0;$$

then both bisectors are represented by the equation

$$x \cos \alpha + y \sin \alpha - p = \pm x \cos \alpha' + y \sin \alpha' - p'.$$

For, each of the two lines this represents must pass through the intersection of the given lines; and as each member of the equation represents the length of the perpendicular from a point which satisfies it on the two given lines, the lengths of these perpendiculars being always equal to each other, the lines fulfilling this condition must be the bisectors of the angles between the given ones.

If the given equations are in form $Ax + By + C = 0$ and $A'x + B'y + C' = 0$, then the equation representing the bisectors will be

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}.$$

The reader will observe, in both cases, that the equation which represents the two bisectors represents two lines which fulfil the condition of being at right angles to each other.

In the latter case, m becomes

$$\frac{B\sqrt{A'^2 + B'^2} + B'\sqrt{A^2 + B^2}}{A\sqrt{A'^2 + B'^2} + A'\sqrt{A^2 + B^2}},$$

and m' becomes

$$\frac{B\sqrt{A'^2 + B'^2} - B'\sqrt{A^2 + B^2}}{A\sqrt{A'^2 + B'^2} - A'\sqrt{A^2 + B^2}},$$

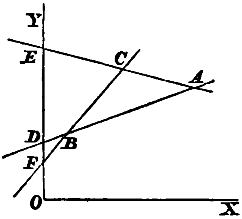
from which it easily follows that

$$m \,.\, m' = \frac{A'^2B^2 - A^2B'^2}{-(A'^2B^2 - A^2B'^2)} = -1.$$

27. *To find the area of a triangle formed by three straight lines whose equations are given.*

Let the side AB be formed by the straight line whose equation is $y = m_1x + c_1$, and AC be formed by the straight line $y = m_2x + c_2$, and BC by the straight line $y = m_3x + c_3$. The area of the triangle ABC

$= $ the triangle EAD
$+$ the triangle DBF
$-$ the triangle ECF
$= \frac{1}{2}ED \times x$ of point A
$+\frac{1}{2}DF \times x$ of point B
$-\frac{1}{2}EF \times x$ of point C



(the abscissæ of the points A, B, and C being respectively the perpendiculars of the triangles whose bases are ED, DF, and EF). But $ED = c_2 - c_1$, *i.e.*, the difference of the intercepts on the axis of y made by the lines $y = m_1x + c_1$ and $y = m_2x + c_2$. Similarly $DF = c_1 - c_3$, and $EF = c_2 - c_3$, and the abscissa of A is found by solving in x the equations $y = m_1x + c_1$ and $y = m_2x + c_2$. Subtract them, and we have

$$(m_1 - m_2)\,x = -(c_1 - c_2)\,; \text{ hence } x = \frac{c_2 - c_1}{m_1 - m_2}.$$

Therefore the triangle

$$EAD = \tfrac{1}{2}ED \times x \text{ of } A = \tfrac{1}{2}(c_2 - c_1) \times \frac{c_2 - c_1}{m_1 - m_2} = \frac{(c_2 - c_1)^2}{2(m_1 - m_2)}.$$

By analogy (for it is not necessary to repeat the work) the triangle $$DBF = \frac{(c_1 - c_3)^2}{2(m_3 - m_1)}\,;$$

and the triangle $$ECF = \frac{(c_2 - c_3)^2}{2(m_3 - m_2)}.$$

Therefore the triangle

$$ABC = \frac{(c_2 - c_1)^2}{2(m_1 - m_2)} + \frac{(c_1 - c_3)^2}{2(m_3 - m_1)} - \frac{(c_2 - c_3)^2}{2(m_3 - m_2)}\,;$$

or, changing the last sign,

$$\frac{(c_2 - c_1)^2}{2(m_1 - m_2)} + \frac{(c_1 - c_3)^2}{2(m_3 - m_1)} + \frac{(c_2 - c_3)^2}{2(m_2 - m_3)}.$$

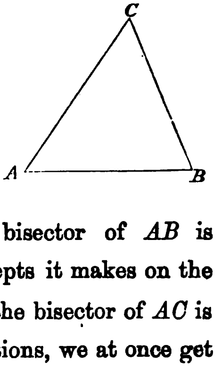
This may be put in the form

$$\frac{\{c_1(m_3-m_2)+c_2(m_3-m_1)+c_3(m_2-m_1)\}^2}{2(m_3-m_2)(m_3-m_1)(m_2-m_1)}.$$

It is obvious that, if the three points are in a straight line, the numerator equals nothing (for $c_1 = c_2 = c_3$), as the three lines must coincide, and so the area vanishes; and if two of the lines are parallel, the denominator becomes nothing, that is, the area becomes infinite.

28. The following examples illustrate how a question may be simplified by a judicious choice of coordinates.

To prove that the three bisectors of the sides of a triangle drawn from the opposite angles meet in a point, and are cut in that point in the ratio of 2 : 1.



Take the angular point A for origin, and the sides AB and AC as axes of x and y. Calling the length of the side BC, a, and of AB and AC respectively c and b, the coordinates of the middle point of BC are manifestly $x = \frac{1}{2}c$, $y = \frac{1}{2}b$, and consequently the equation to the bisector of BC is $bx = cy$ (Art. 20).

By Art. 11, the equation to the bisector of AB is $\frac{2x}{c} + \frac{y}{b} = 1$, ($c$ and b being the intercepts it makes on the axes of x and y,) and the equation to the bisector of AC is $\frac{x}{c} + \frac{2y}{b} = 1$. Solving these two equations, we at once get $y = \frac{1}{3}b$ and $x = \frac{1}{3}c$, which values satisfy the equation to the bisector of BC; and also (by Art. 7) prove that the lines are cut in the ratio stated.

29. *The three perpendiculars from the angular points on the sides of a triangle meet in a point.*

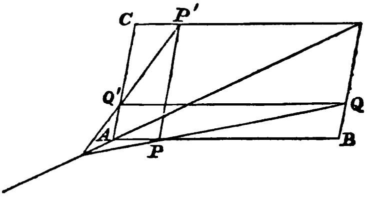
Take A as origin, AB as axis of x, and a straight line perpendicular to it at A as axis of y. Let the coordinates of C be x', y'; and let $AB = c$. The equation to AC is

$y = \frac{y'}{x'}x$; and to the perpendicular on it from B $(x=c, y=0)$ is $y = \frac{-x'}{y'}(x-c)$. The equation to BC, joining the points $(c, 0)$ and (x', y'), is $y = \frac{y'}{x'-c}(x-c)$; and the equation to the perpendicular on it from the origin is $y = \frac{-(x'-c)}{y'}x$. Equating these two values of y, we get the x of the point of intersection of these two perpendiculars from

$$\frac{(x-c)}{y'}x = \frac{x'(x-c)}{y'}.$$

The value of x here is manifestly x', as it makes both sides identical. Consequently the ordinate of the point of intersection must coincide with y', the ordinate of C, which is also the perpendicular from it on AB.

30. *To prove that, if two parallels be drawn to the sides of a parallelogram, the straight lines joining the points in which they meet the sides will intersect on the diagonal.*



Let PP' and QQ' be the two parallel lines. Take two sides of the parallelogram AB and AC as axes, let their length be a, b, and let $AQ' = m$, and $AP = n$. Then the equation to PQ joining P $(n, 0)$ to Q (a, m) is $\frac{y}{x-n} = \frac{m}{a-n}$, or $ay - ny = mx - mn$. Likewise the equation to $P'Q'$ joining P' $(n, 0)$, to Q' $(0, m)$ is $\frac{y-b}{x-n} = \frac{m-b}{-n}$; or, $bn - ny = mx - bx - mn + bn$. Subtracting these two equations, we get $ay = bx$, which must be a line through their intersection (Art. 25), whatever the values of m and n, and which is the equation of the diagonal through A. This result, which is here so easily obtained, would present much difficulty to Plane Geometry.

EXERCISES.

1. Find the equations of the straight lines joining the points (i.) (2, 1) and (3, −2); (ii.) (2, 3), and (4, −5); (iii.) (2, 5) and (0, −7).
[*Ans.* (i.) $3x+y=7$; (ii.) $x+y=11$; (iii.) $y-6x+7=0$.

2. Find the coordinates of the angles of the triangle whose sides are represented by the equations $3x+y=2$, $x+2y=5$, $2x-3y+7=0$.
[*Ans.* $(\frac{1}{5}, \frac{13}{5})$, $(-1, 3)$ $(-\frac{1}{11}, \frac{25}{11})$.

3. Find the equation to the straight line which passes through the point (1, 3), and makes an angle of 30° with the straight line $2y-x=5$.
[*Ans.* $11y-(8+5\sqrt{3})\,x=5\,(5-\sqrt{3})$.

4. Find the equation to the straight line which is perpendicular to the straight line $4y+5x=21$, and cuts the axis of y at a distance $=3$ from the origin.
[*Ans.* $5y-4x=15$.

5. Find the angle between the two straight lines whose equations are $3x+2y+1=0$, $2x-3y=7$. [*Ans.* 90°.

6. Find the angle between the lines $2x+3y=4$, $3x+4y+7=0$. [*Ans.* $\tan^{-1}\frac{1}{18}$.

7. Find the equation to the bisector of the angle between the two lines in last example.
[*Ans.* $(10-3\sqrt{13})\,x+(15-4\sqrt{13})\,y=20+7\sqrt{13}$.

8. Find the length of the perpendicular from the point (5, 6) on the straight line $3x+4y-9=0$. [*Ans.* 6.

9. Find the area of the triangle formed by the straight lines $x+2y=5$, $2x+y=7$, $y-x=1$. [*Ans.* $\frac{3}{2}$.

10. Find the area of the triangle formed by the straight lines $y=2x+3$, $y=3x+4$, $y=4x+5$.
[*Ans.* 0, ∴ they meet in a point.

11. Given base and difference of squares of sides of a triangle, find equation to the locus of its vertex.

12. A straight line is drawn parallel to the base of a triangle, and the points where it cuts the sides joined with the opposite angles, find the locus of the intersection of the joining lines.

[*Ans.* $\frac{x}{a}=\frac{y}{b}$, where the sides a, b are the axes.

13. Given the vertical angle and the sum of the sides in a triangle, find the locus of a point where the base is cut in a given ratio.

[*Ans.* $x+y=\frac{km}{1+k}$, where m is the given sum and k the given ratio, the axes being the same as in last example.

14. Shew that the straight lines represented by the equations $\frac{m}{r}=4\cos\theta+3\sin\theta$, and $\frac{n}{r}=3\cos\theta-4\sin\theta$, are at right angles.

15. What does $\sin 3\theta=0$ represent? In general what does any function of $\theta=0$ represent?

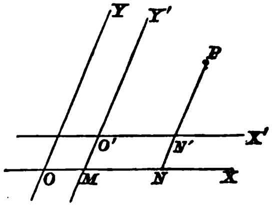
[*Ans.* Three straight lines through the origin inclined at the angles 0°, 60°, and 120° to the initial line. $f\theta=0$ represents straight lines through the origin.

CHAPTER IV.

TRANSFORMATION OF COORDINATES.

It is sometimes very important to change an equation from one set of coordinates to another, that is, to find what the equation becomes when referred to another origin and axes in the same plane.

31. *To transform an equation from one pair of axes to another in the same direction.*



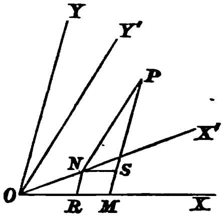
Let the coordinates of the new origin (O') referred to the original axes be a, b. Take any point in the plane, and the abscissæ referred to the original axes will exceed the abscissæ referred to the new axes by a, and the ordinate referred to the original axes will exceed the ordinate referred to the new axes by b. Thus, let P be the point, let $OM = a$, and $MO' = b$ (the coordinates of the new origin O'), then ON, the abscissa on x of P, referred to the original axes, is equal to $OM + MN = OM + O'N'$; that is, it is equal to a, the abscissa of the new origin, $+$ the abscissa of the point P referred to the new origin. Similarly $PN = PN' + N'N = PN' + O'M$, or the ordinate to the old axes equals the ordinate to the new axes $+b$.

Therefore, to transform an equation to a pair of axes in the same direction, but intersecting in an origin whose coordinates to the given axes are a, b,—for x substitute $x + a$, and for y substitute $y + b$.

The resulting equation will represent the locus referred to the new axes.

This applies equally to rectangular and oblique coordinates.

32. *To transform an equation from one pair of axes to another with the same origin, but different directions.*



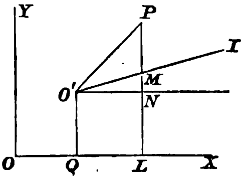
Let both the axes be oblique, and let XOY be the old axes, the angle between them being w; and let $X'OY'$ be the new axes; also let XOX', the angle the new axis of x makes with the old axis of x, be α; and XOY', the angle the new axis of y makes with the old axis of x, be β. Drawing the coordinates of any point P, we have OM and PM the coordinates of P to the old axes; and ON and PN the coordinates of P to the new axes. It is required to express OM and PM in terms of ON and PN. Draw NS and NR parallel to the original axes OX, OY. Then $OM = OR + RM$. But, from the triangle ORN, $OR = ON \times \dfrac{\sin RNO}{\sin ORN}$; and from the triangle NSP, we have $NS\ (= RM) = NP\dfrac{\sin SPN}{\sin NSP}$. Therefore we find $OM = ON\dfrac{\sin RNO}{\sin ORN} + NP\dfrac{\sin SPN}{\sin NSP}$; or, x referred to old axes equal to $x\dfrac{\sin(w-\alpha)}{\sin w} + y\dfrac{\sin(w-\beta)}{\sin w}$ referred to new axes. Similarly we have $PM = SP + SM = SP + NR$, or, taking the values from the same triangles, y referred to old axes, $PM = PN\dfrac{\sin SNP}{\sin NSP} + ON\dfrac{\sin NOR}{\sin ORN}$, or

$$y = y\frac{\sin\beta}{\sin w} + x\frac{\sin\alpha}{\sin w}$$

referred to new axes. Therefore to transform the equation we must substitute for x, $x\dfrac{\sin(w-\alpha)}{\sin w} + y\dfrac{\sin(w-\beta)}{\sin w}$; and for y, $y\dfrac{\sin\beta}{\sin w} + x\dfrac{\sin\alpha}{\sin w}$. If the original axes are rectangular, then $\sin w = 1$ and vanishes as a denominator; and the value of x becomes $x\cos\alpha + y\cos\beta$; and the value of y becomes $y\sin\beta + x\sin\alpha$.

If both the axes are rectangular, then we have also $\beta = \frac{1}{2}\pi + \alpha$; and x becomes $x \cos \alpha - y \sin \alpha$; and y becomes $y \cos \alpha + x \sin \alpha$.

33. *To transform an equation from rectangular to polar coordinates.*



Let O be the origin, and OX and OY the rectangular axes; let O' (whose coordinates are a, b) be the origin for polar coordinates, and $O'I$ the initial line, and let the angle $IO'N$ it makes with the axis of x be α. It is required to express the abscissa OL, and ordinate PL, of any point P in terms of the radius vector $O'P = r$, and the vectorial angle $IO'P = \theta$.

$$OL = OQ + QL = OQ + O'N = OQ + O'P \cos NO'P,$$

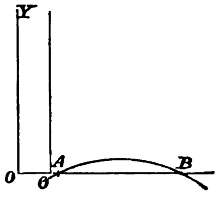
that is, $x = a + r \cos(\theta + \alpha)$. Similarly, we have

$$PL = LN + PN = QO' + O'P \sin NO'P,$$

that is, $y = b + r \sin(\theta + \alpha)$. These values substituted for x and y change the equation from rectangular to polar coordinates.

If the initial line is parallel to the axis of x, then the angle α vanishes.

34. It is manifest that these transformations do not alter the degree of the equation. For, in place of x and y, x or y ± a constant, or multiplied by some known factor, must be substituted. Therefore the unknown quantity must remain of the same degree as before.



35. It may now be easily shewn that an equation of the first degree must represent a straight line. Suppose that the locus represented by a simple equation

$$Ax + By + C = 0$$

was not a straight line; but a line (AB) such that two points[*] in it

A and B could be joined by a straight line which did not coincide with the locus of the equation. Transform the origin to any point (O) on the straight line AB, and make AB the axis of x, and a straight line perpendicular to it the axis of y. Now, for $y = O$, x must have two values, OA and OB, therefore the curved line cannot be represented by a simple equation; and it has been before shewn that an equation always represents some line; that is, a series of points. (Art. 10.)

Sometimes an equation of the second or of a higher degree can be broken into factors, that is, into two or more simple equations as many as the degree of the equation. In that case the equation represents as many straight lines as it has factors that are simple equations, and each straight line is represented by one of these simple equations.

For example, suppose that

$$Ax^2+Bxy+Cy^2+Dx+Ey+F = 0,$$

was the product of

$$a_1x+b_1y+c_1 = 0 \text{ and } a_2x+b_2y+c_2 = 0,$$

then the quadratic would represent the two straight lines which are represented respectively by the two simple equations. For, take any value of x, and put it in the above quadratic, there will be two values of y corresponding to it; for it will become an ordinary quadratic with one unknown quantity, thus

$$A'+B'y+Cy^2+D'+Ey+F = 0$$

(the dotted letters include the given value of x, multiplying the coefficients).

But since it is the product of the two simple equations

$$a_1+b_1y+c_1 = 0 \text{ and } a_2+b_2y+c_2 = 0,$$

they must give the same two values of y. Therefore for any value of x there are two values of y, one of which gives a point on one straight line, and the other gives a point on the second straight line. Should the value of x be the abscissa of the intersection of the two straight lines, then there will be two equal values of y which will be the ordinate of the point of intersection.

Again, the equation may have no real roots; in that case no real locus is represented by it, as there are no real values of x and y.

Again, an equation may have only one possible pair of values for x and y. In that case it represents the point which has those values of x and y for its coordinates. Thus $(x-a)^2+(y-b)^2=0$ can have only one value for x and y respectively, that is, $x=a$ and $y=b$, consequently it represents the point $x=a$, $y=b$.

36. An equation containing only x represents a series of lines parallel to the axis of y. For it can be put in the form $(x-\alpha), (x-\beta), (x-\gamma)$ &c. $=0$, which is satisfied by $x=\alpha$, $x=\beta$, &c.

In the same way, an equation in y only represents a series of lines parallel to the axis of x.

An equation in the form of a function of $\frac{x}{y}$, as

$$A\left(\frac{x}{y}\right)^n+B\left(\frac{x}{y}\right)^{n-1}+\dots K=0,$$

represents a series of lines passing through the origin. For it is equivalent to

$$\left(\frac{x}{y}-\alpha\right)\left(\frac{x}{y}-\beta\right)\left(\frac{x}{y}-\gamma\right)\text{ \&c.}=0,$$

which is satisfied by $x=\alpha y$ and $x=\beta y$, &c., each of which represents a straight line passing through the origin.

37. When it is required to break up an equation of the second degree into two simple equations, this can generally be done by one or two trials (when it is known that the equation is capable of being so broken up), remembering that the indeterminate coefficient is the product of the two indeterminate coefficients in the simple equations, the coefficient of x^2 is the product of the coefficients of the x, the coefficient of y^2 the product of the coefficients of y, and the coefficient of xy the sum of the products of the coefficients $x\times y$ and $y\times x$ in the two simple equations, while the coefficients of x and y in the quadratic are the sum of the products of the indeterminate coefficients in one equation

and the coefficients of x or y in the other. A few trials with some of the coefficients will enable us in general to guess what they must be in the required simple equations. It must be borne in mind that the sum means the algebraical sum, in which the signs as well as the amounts are taken into consideration. The rule may be stated thus; if

$$Ax^2+Bxy+Cy^2+Dx+Ey+F = (ax+by+c)\times(a'x+b'y+c'),$$

then $A=aa'$, $B=ab'+a'b$, $C=bb'$, $D=ac'+a'c$,

$E=bc'+b'c$, and $F=cc'$.

38. The general condition that

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

should break up into two simple equations may be found thus:—

Treat the equation as a quadratic in y, thus

$$Cy^2+(Bx+E)\,y+Ax^2+Dx+F=0.$$

The roots of this will be

$$y=-\frac{Bx+E}{2C}\pm\frac{\sqrt{(Bx+E)^2-4C\,(Ax^2+Dx+F)}}{2C}.$$

If the expression under the radical sign is a perfect square, then the equation manifestly breaks up into two simple equations; and the condition that the expression under the radical sign should be a perfect square, is

$$(BE-2CD)^2=(B^2-4AC)\,(E^2-4CF).$$

Examples.

1. Transform the equation $x^2+y^2-4x-6y=18$ to the point (2, 3) as origin. [*Ans.* $x^2+y^2=31$.

2. Transform the equation $x^2-y^2=6$ to axes bisecting the angle between the given ones. [*Ans.* $xy=3$.

3. Transform the equation $r^2=m^2\cos 2\theta$ to rectangular axes. [*Ans.* $(x^2+y^2)^2=(x^2-y^2)\,m^2$.

4. Transform the equation $x^4+y^4+6a^2y^2=2$ to another pair of rectangular axes inclined at an angle of 45° to the given axes. [*Ans.* $x^4+y^4=1$.

5. Transform $y^2+4ay\cot\alpha-4ax=0$ from rectangular axes to a pair inclined at the angle α, the origin and axis of x being the same. [*Ans.* $y^2=\frac{4ax}{\sin^2}$

6. Shew that the equation $2y^2-3xy-2x^2+2x+y=0$ represents two straight lines at right angles to each other.

7. What straight lines are represented by the equation $y^2-xy-2x^2+5x-y=2$; and what is the angle between them?

[*Ans.* $y-2x+1=0$, and $y+x-2=0$, angle $=\tan^{-1}3$.

8. Find the straight lines, and the angle between them, represented by the equation $5y^2+xy-x^2=0$.

[*Ans.* $2y+x=0$, $3y=x$; angle 135°.

9. Find what are expressed by the equations:

(i.) $x^2+y^2=0$; (ii.) $xy=0$; (iii.) $(x-a)(y-b)=0$;
(iv.) $x^2-ax=0$; (v.) $x^2-xy=0$.

[*Ans.* (i.) the origin; (ii.) the axes; (iii.) the straight line $x=a$, $y=b$; (iv.) the axis of y, and the line $x=a$; (v.) the axis of y and the line bisecting the angle between the axes.

10. What are represented by

(i.) $x^2-y^2=0$; (ii.) $(x-a)^2+(y-b)^2=0$;
(iii.) $(x-y+a)^2+(x+y-a)^2=0$; (iv.) $x^2+y^2+a^2=0$.

[*Ans.* (i.) the line bisecting the angles at the origin; (ii.) the point a, b; (iii.) the point 0, $-a$; (iv.) impossible.

CHAPTER V.

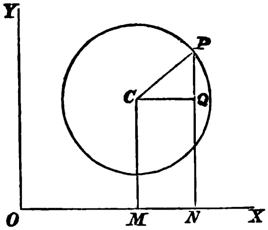
THE CIRCLE.

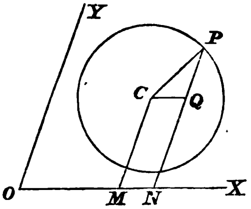
39. We will now proceed to find the equation which represents a circle, the coordinates of the centre and length of the radius being given.

Let the coordinates of the centre be $x=a$, $y=b$, and the length of the radius be called r. Then take any point P on circumference whose coordinates are x, y, and they can be thus connected with the given coordinates of the centre C and the length of the radius. Let the axes be oblique, and the angle between them be called w.

Draw the ordinates C and P, CM and PN, and CQ parallel to the axis of x. Then

$$CP^2 = CQ^2+PQ^2-2CQ\,.\,PQ\cos CQP,$$





that is, $r^2 = (x-a)^2+(y-b)^2+2\,(x-a)\,(y-b)\cos w$, for $CQP = 180^\circ-w$. Expanding this, it becomes

$$x^2+y^2+2xy\cos w-2\,(a+b\cos w)\,x$$
$$-2\,(b+a\cos w)\,y+a^2+b^2+2ab\cos w = r^2.$$

Hence the general form of the equation to a circle referred to oblique axes is

$$x^2+y^2+2xy\cos w+Ax+By+C=0.$$

40. When the axes are rectangular, $w = 90°$, therefore $\cos w = 0$, and the coefficient of xy vanishes. The form of the equation to a circle referred to rectangular axes consequently is $x^2+y^2+Ax+By+C = 0$.
This follows at once from the figure

$$CP^2 = CQ^2+PQ^2, \text{ that is, } r^2 = (x-a)^2+(y-b)^2,$$

which becomes $x^2+y^2-2ax-2by+a^2+b^2-r^2 = 0$.

From this it follows that, if an equation is of the form

$$x^2+y^2+Ax+By+C = 0,$$

it represents a circle, the coordinates of whose centre are $x = -\frac{1}{2}A$, $y = -\frac{1}{2}B$, and its radius is $(\frac{1}{4}A^2+\frac{1}{4}B^2-C)^{\frac{1}{2}}$. For $C = a^2+b^2-r^2 = \frac{1}{4}A^2+\frac{1}{4}B-r^2$.

If the axes are oblique, then there are three equations to determine the coordinates of the centre and the length of the radius, $a+b\cos w = -\frac{1}{2}A$, $b+a\cos w = -\frac{1}{2}B$, $a^2+b^2+2ab\cos w-r^2 = C$, in which a, b, and r are the unknown quantities; and if w, the angle between the axes, should not be given, it can be found from the coefficient of xy, which $= 2\cos w$.

41. We will now consider some particular forms of the equation to the circle referred to rectangular axes.

If the origin is on the circumference, then $C = 0$, for the equation must be satisfied with the values $x=0$, $y=0$; and also it is evident that $r^2 = a^2+b^2$.

If the axis of x passes through the centre, then $b=0$, and the coefficient of y vanishes; if the axis of y passes through the centre, $a = 0$, and the coefficient of x vanishes.

Consequently, if the centre be the origin, the equation to the circle becomes $x^2+y^2 = r^2$. This is very evident from the figure.

The equation to a circle can always be reduced to this, which is its simplest form, by transforming the origin to a point $x = -\frac{1}{2}A$, $y = -\frac{1}{2}B$; the direction of the axes being unchanged.

42. Taking this simplest form of the equation, we will

now proceed to find the equation of the tangent to the circle represented by $x^2+y^2=r^2$.

Let (x', y') be any point on the circle, and x'', y'' any other point adjacent to it and also on the circle. The line joining them is a secant of the circle, and its equation is

$$y-y'=\frac{y''-y'}{x''-x'}(x-x').$$

Now the tangent may be considered as a particular case of a secant, viz., when the two points in which a straight line cuts a circle are infinitely near each other. We may suppose one point to be fixed, and the line moved round it, bringing the other point nearer to it, until it coincides; then manifestly the line touches the circle. (N.B.—This definition of the tangent is applicable to the case of any curve and not peculiar to the circle.) Applying this to the equation

$$y-y'=\frac{y''-y'}{x''-x'}(x-x'),$$

when the points (x', y') and (x'', y'') coincide, $x'=x''$ and $y'=y''$. This renders the equation indeterminate, as $\frac{y''-y'}{x''-x'}=\frac{0}{0}$. But, by taking into consideration that both the points (x', y') and (x'', y'') are on the circle, a definite value can be found for the fraction $\frac{y''-y'}{x''-x'}$.

Since (x', y') is a point on the circle, $x'^2+y'^2=r^2$; and since (x'', y'') is also on the circle, $x''^2+y''^2=r^2$. Subtracting, we have $x'^2-x''^2+y'^2-y''^2=0$; or

$$(x'+x'')(x'-x'')+(y'+y'')(y'-y'')=0;$$

therefore $$\frac{y'-y''}{x'-x''}=-\frac{x'+x''}{y'+y''}.$$

Therefore the equation to the secant of a circle may be written thus, $$y-y'=-\frac{x'+x''}{y'+y''}(x-x'),$$

when (x', y') and (x'', y'') are the two points on the circle through which it passes. And in this form it does not become indeterminate when $x'=x''$ and $y'=y''$. Therefore the equation to the tangent to the circle is

$$y-y'=-\frac{2x'}{2y'}(x-x'),$$

when x' and y' are the coordinates of the point of contact. This becomes $\quad yy'-y'^2=-xx'+x'^2,$

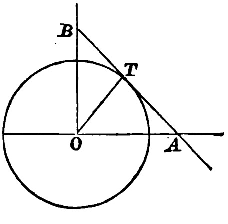
or $\quad xx'+yy'=x'^2+y'^2=r^2.$

43. From this it follows that the perpendicular from the centre, which is the origin, on the tangent is the radius. For, by Art. 24, the perpendicular from the origin on the straight line $Ax+By+C$ is $\frac{-C}{\sqrt{A^2+B^2}}$; therefore perpendicular from origin on $xx'+yy'-r^2=0$ is

$$\frac{r^2}{\sqrt{x'^2+y'^2}}=\frac{r^2}{r}=c.$$

Also, the equation to the radius drawn to the point of contact is $y'x=x'y$, which is manifestly the equation to a straight line perpendicular to $xx'+yy'=r^2$.

If from the equations $x^2+y^2=r^2$, and $xx'+yy'=r^2$, we find the values of x and y, they get two equal values of x, $x=x'$, and two equal values of y, $y=y'$, which shows that the straight line $xx'+yy'=r^2$ meets the circle $x^2+y^2=r^2$ in one point only, and therefore, by Euclid, Book III., Prop. 16, is a tangent.



Any line whose equation is of the form $y=mx+r\sqrt{1+m^2}$, touches the circle $x^2+y^2=r^2$. For it is evident from the figure that the intercept (OB) made on the axis of y is equal to OT sec TOB $=OT$ sec TAO (OT being the perpendicular from origin on the straight line)

$$=OT\sqrt{1+\tan^2 TAO}=OT\sqrt{1+m^2}.$$

But $OB=r\sqrt{1+m^2}$, therefore $OT=r$. Consequently a circle with centre at O and radius equal to r touches the straight line AO, whose equation is $y = mx + r\sqrt{1+m^2}$.

44. *To find the equation of the tangent to a circle whose equation is of the form* $(x-a)^2+(y-b)^2 = r^2$.

Let x', y', and x'', y'', be the coordinates of two adjacent points on the circle, the equation to the line joining them is

$$y - y' = \frac{y''-y'}{x''-x'}(x-x').$$

Since the points are on the circle, we have

$$(x'-a)^2+(y'-b)^2 = c^2 \text{ and } (x''-a)^2+(y''-b)^2 = r^2;$$

subtracting,

$$(x''-a)^2-(x'-a)^2+(y''-b)^2-(y'-b)^2 = 0,$$

or $(x''-x')(x''+x'-2a)+(y''-y')(y''+y'-2b) = 0$;

from which it follows that $\dfrac{y''-y'}{x''-x} = \dfrac{x''+x'-2a}{y''+y'-2b}$. The equation to the secant may therefore be written

$$y - y' = \frac{x''+x'-2a}{y''+y'-2b}(x-x'),$$

which, when $x' = x''$ and $y' = y''$, and the line is tangent, becomes $y - y' = \dfrac{x'-a}{y'-b}(x-x')$. This can be put into a convenient form thus,

$$y-b-(y'-b) = \frac{x'-a}{y'-b}\{x-a-(x'-a)\};$$

and, multiplying up,

$$(x-a)(x'-a)+(y-b)(y'-b) = (x'-a)^2+(y'-b)^2 = r^2.$$

45. *To find the equation of the chord of contact of tangents drawn to a circle from a given external point.*

Let the coordinates of the given external point be h, k, then a tangent drawn from it to the circle will have for its equation $xx' + yy' = r^2$, where x' and y' are the coordinates of the point of contact. But since this tangent also passes through the point (h, k), its equation must be satisfied by the values $x = h$ and $y = k$. Therefore

$$hx' + ky' = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, if the coordinates of the other tangent drawn to the circle from the point (h, k), are x'', y'', then its equation is $xx'' + yy'' = c^2$, and this must also be satisfied by the values h and k, therefore $hx'' + ky'' = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$. From this it follows that $hx + ky = r^2$ is the equation of the chord of contact. For the straight line represented by this equation passes through the point (x', y'), as, by (1), these values of x and y satisfy it. Similarly, it must, by (2), pass through (x'', y''). Therefore it is the straight line joining the two points of contact (x', y') and (x'', y'').

From this it follows that, if the coordinates of any external point be put into the equation of the tangent instead of the coordinates of the point of contact, the resulting equation is that of the chord of contact of the tangents drawn from the external point.

46. *If from any fixed point chords be drawn to a circle, the tangents to the circle at the extremities of any such chord will intersect on a fixed straight line.*

Let h, k be the coordinates of the point, and let (x', y') be the point at which a pair of tangents drawn at the extremities of a chord through (h, k) intersect. Now the chord of contact to the point (x', y') is $xx' + yy' = r^2$. But, by supposition, it passes through (h, k); therefore $hx' + ky' = r^2$. This condition holds for any other point (as well as for x', y'), which is the intersection of tangents drawn at the extremities of any chord through (h, k). Consequently $hx + ky = r^2$ must represent a straight line on which the tangents drawn at the extremities of any chord through (h, k) must intersect.

47. The reader will observe the position of this straight line as the point (h, k) is inside or outside the circle. If it is taken outside the circle, then the property may be thus stated. If chords be drawn to a circle from any external point, the tangents at the extremities of any such chord intersect on the chord of contact of the external point. In

this instance we see how a theorem, which could not easily be proved by purely geometrical methods, follows at once from the equations which represent the lines in Analytic Geometry.

48. *If points be taken on a given straight line, the chords of contact will pass through a fixed point.*

Let the equation to the given straight line be

$$Ax + By + C = 0,$$

and let the coordinates of a point in it be x', y', the equation of its chord of contact is $xx' + yy' = r^2$. But since (x', y') is on the given straight line, we have $Ax' + By' + C = 0$, or $y' = -\dfrac{Ax' + C}{B}$.

Putting in this value of y' in the equation to the chord of contact, it becomes

$$xx' - y\,\frac{Ax' + C}{B} = r^2, \text{ or } (Bx - Ay)\,x' - (Cy + r^2) = 0.$$

This is in the form $L - kL'$, the constant being x'; therefore, by Art. 25, this represents a straight line which passes through the intersection of the straight lines $Bx - Ay = 0$, $Cy + c^2 = 0$, whatever be the value of x'.

POLE AND POLAR.

49. The general relation between a point and its chord of contact is expressed by the terms *pole* and *polar*, the chord of contact being called the *polar* of the point, and the point the *pole* of its chord of contact.

This use of the terms must not be confounded with polar coordinates, or the polar equations to a line or circle.

Thus, with reference to the circle $x^2 + y^2 = r^2$, the straight line $hx + ky = r^2$ is the polar of the point (h, k); and the point (h, k) is the pole of the straight line $hx + ky = r^2$.

The properties of poles and polars may be thus expressed.

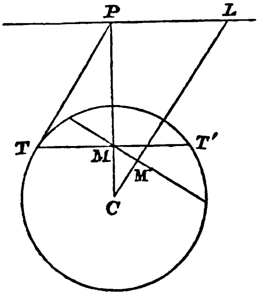
If points lie on a straight line, their polars will pass through a fixed point (*i.e.*, the pole of the straight line).

If straight lines pass through a given point, their poles will be on a straight line (*i.e.*, the polar of the given point).

These properties follow immediately from the three preceding articles; but it may be well to give a geometrical definition and proof of the properties of poles and polars.

If a perpendicular be drawn from the centre of a circle to a straight line, and be cut or produced so that the rectangle under the segments from the centre is equal to the square of the radius, the point so found is called the *pole* of the straight line.

If a given point be joined with the centre of a circle, and the joining line cut or produced so that the rectangle under its segments measured from the centre is equal to the square of the radius, the perpendicular to the joining line at that point is called the polar of the given point.



It is evident, if the point is outside the circle, that the polar is its chord of contact. For, let P be the point and TM its polar, according to definition, (*i.e.*, $CM \times CP = r^2$,) then PT will be a tangent to the circle. For a circle on PT as diameter will pass through M (TMP being a right angle); and since $CT^2 = CM \times CP$, CT must be a tangent to it (Euc. Book III., 37). Therefore the angle PTC is right; therefore PT is a tangent to the circle whose centre is at C. Similarly PT' is a tangent.

If any point (L) be taken on the straight line PL at right angles to CP, its polar will pass through (M), the pole of PL. Join CL, and from M draw MM' perpendicular to CL; then MM' is the polar of the point L. For, since the angles at P and M' are right, a circle can be described round the quadrilateral $MPLM'$. Therefore

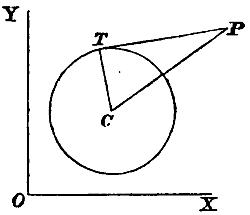
$$CM' \times CL = CM \times CP = r^2;$$

therefore MM' is the polar of L.

The converse of this is evidently proved in the same manner.

By the use of poles and polars, many theorems about straight lines meeting in a point can be made to prove corresponding theorems about points lying on a straight line, and *vice versâ*.

50. *To find the length of the tangent drawn from any point to a circle.*



Let the equation of the circle be $(x-a)^2+(y-b)^2-r^2=0$, and the coordinates of the point P be x', y'. Now the square of the straight line PC, joining the point with the centre of the circle C, is equal $(x'-a)^2+(y'-b)^2$ (Art. 6). But the square of the tangent from P is equal to the difference between the square of PC and the square of the radius. That is,

$$PT^2 = PC^2 - CT^2 = (x'-a)^2+(y'-b)^2-r^2;$$

that is, the square of the tangent from any point is found by putting the coordinates of the point in the equation of the circle.

The interpretation of this with reference to the equation of the circle may be stated thus. The length of the tangent from any point on the circumference is nothing; therefore $(x-a)^2+(y-b)^2-r^2=0$ is true for any point on the circumference.

If the point (P) be inside the circle, then $CP^2<r^2$; consequently $(x'-a)^2+(y'-b)^2-r^2$ is a negative quantity whose square root is impossible; that is, no tangent can be drawn to a circle from a point inside it.

If any chord be drawn through P, the rectangle under the segments of that chord

$$= CP^2-c^2 = (x'-a)^2+(y'-b)^2-r^2.$$

This, when the point is external, is a positive, that is, the segments of the chord are measured in the *same* direction from P. When P is inside the circle, then

$$(x'-a)^2+(y'-b)^2-r^2$$

is a negative quantity, and is equal to the rectangle under the segments of the chord, which are measured in *opposite* directions from P. This agrees with the principle laid down in Article 3, that straight lines measured in opposite directions have opposite signs.

51. *To find the radical axis of two circles.*

(The straight line which is the locus of points from which equal tangents can be drawn to two circles is called their *radical axis.*)

If the two circles intersect, this locus is manifestly their common chord; and if they do not intersect, it is easily found by Plane Geometry. But it can be found at once by Analytic Geometry thus. Let the equation of one circle be $x^2+y^2+Ax+By+C=0$, and of the other circle

$$x^2+y^2+A'x+B'y+C'=0;$$

then, subtracting them, we get

$$(A-A')\,x+(B-B')\,y+C-C'=0,$$

which is the equation of their radical axis. For, it is the equation to some straight line, as it contains only the first power of x and y; and the coordinates of any point on it (x',y'), if put into the equations of the two circles, make them equal; therefore the squares of the tangents from this point to the two circles are equal, therefore the tangents are equal from any such point.

It may be well to explain this more clearly. The equation $(A-A')\,x+(B-B')\,y+C-C'=0$ may be written

$$x^2+y^2+Ax+By+C-(x^2+y^2+A'x+B'y+C')=0.$$

Therefore x', y', the values which satisfy the equation, make

$$x'^2+y'^2+Ax'+By'+C=x'^2+y'^2+A'x'+B'y'+C'.$$

But the two sides of this equation represent respectively the squares of the tangents from the point (x', y') to the two circles; therefore the tangents are equal.

52. If for shortness the equation to the circle

$$x^2+y^2+Ax+By+C=0$$

be written $S=0$, and the equation to the second circle

$S'=0$, then the equation to their radical axis is $S-S'=0$. From this it is evident that the radical axes of three circles meet in a point. For the equation to the radical axis of the first and second is $S-S'=0$, and the equation to the radical axis of the second and third is $S'-S''=0$, and the equation to the radical axis of the third and first is $S''-S=0$, since the sum of these equations equals nothing; or, as any one of them is the difference of the other two, the three lines they represent must meet in a point, by Art. 25. This point is called their *radical centre.*

If one of the circles becomes a point, *i.e.*, if its radius becomes nothing, the radical axis becomes the locus of points from which tangents to a given circle are equal to the distance from a given point. To this case the foregoing articles are applicable, as a point can be expressed by a quadratic equation in x and y which has only one possible pair of values for x and y. Thus $(x-a)^2+(y-b)^2=0$ represents the point $(x=a, y=b)$; for it can be satisfied by no other values of x and y. If this equation be subtracted from the equation of a circle, the second powers of x and y will vanish, and the resulting simple equation represents the radical axis of the circle and point.

If both the circles become points, the radical axis is the locus of points equally distant from them, that is, manifestly, the perpendicular bisecting the line joining them. The radical centre proves that the three straight lines perpendicular to the sides of a triangle at their middle points meet in a point. For that point may be considered as the radical centre of the three angular points.

The following case of two theorems, which could not be proved without much difficulty by Plane Geometry, is a good instance of the use of abbreviated analytical methods, and of poles and polars.

If in a triangle the angles be joined with the points in which the inscribed circle touches the sides, the joining lines will meet in a point.

Take the centre of the inscribed circle as origin, and any two straight lines at right angles through it as axes. Let the points of contact be x_1y_1, x_2y_2, and x_3y_3; then the equations to the sides will be respectively

$$x_1x+y_1y = r^2 \ldots\ldots(1),\quad x_2x+y_2y = r^2\ldots\ldots\ldots(2),$$

and

$$x_3x+y_3y = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The straight line joining the intersection of (1) and (3) with the point of contact x_2y_2 will be of the form

$$x_1x+y_1y-c^2 = k\,(x_3x+y_3y-c^2),$$

hence it passes through x_2y_2; the value of k is found, by putting in these for x and y in this equation, to be $\frac{x_1x_2+y_1y_2-c^2}{x_3x_2+y_3y_2-c^2}$. Let the equation to the tangents passing through the points x_1y_1, x_2y_2, x_3y_3 be termed respectively T_1, T_2, and T_3; and when x_2, y_2 are substituted for x and y, in (1), let it be called $T_{1,3}$, and let $T_{2,3}$ signify that x_2, y_2 have been substituted for x and y in T_3; then the equation to the straight line joining x_2y_2 with the opposite angles may be written $T_1 = \frac{T_{1,2}}{T_{2,3}}\,T_3$, or $T_1\times T_{2,3} = T_3\times T_{1,2}$.

It is evident that, by the same reasoning, we find the equation of the straight line joining x_3y_3 with the opposite angle is $T_2\times T_{1,3} = T_1\times T_{2,3}$. Also that of the straight line joining x_1y_1 with the opposite angle is

$$T\times T_{1,2} = T_2\times T_{1,3},$$

Since the sum of these equations is equal to nothing, the three straight lines they represent must pass through the same point. Q. E. D.*

If the points in which the inscribed circle touches the sides of a triangle be joined, and the joining lines be produced to intersect the opposite sides, the three points of intersection are in directum.

Let the angle opposite x_1y_1 be A; the angle opposite x_2y_2 be B; and the angle opposite x_3y_3 be C. Then the line joining x_2y_2 with x_3y_3 is the polar of A; and since T_1 is the tangent at x_1y_1, it is the polar of that point; consequently the intersection of these two straight lines is the pole of

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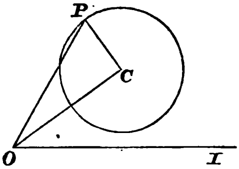
[*] It will be observed that $T_{1,2}$ contains no variables, only x_1y_1 and x_2y_2, therefore it is a constant; and so with the others.

the line joining them. In like manner the other points of intersection are the poles respectively of the straight lines joining B and x_2y_2, and C and x_3y_3. But since (by preceding theorem) these three lines pass through the same point, their poles must lie on the same straight line. Q. E. D.

THE POLAR EQUATION TO THE CIRCLE.



53. The polar equation to the circle is easily found. Let O be the origin, OI the initial line, C the centre of the circle, and P any point on its circumference, the coordinates of it being r and θ, which are variable. Let $OC = l$, and $IOC = a$, so that the coordinates of the centre l, a are given; and let the diameter of the circle be d; so that the radius $CP = \frac{1}{2}d$.

Then $CP^2 = OP^2 + OC^2 - 2\,OP\,.\,OC \cos POC$;

that is, $\frac{1}{4}d^2 = r^2 + l^2 - 2rl\cos(\theta - a)$;

or $r^2 - 2rl(\cos\theta\cos a + \sin\theta\sin a) + l^2 - \frac{1}{4}d^2 = 0$.

This may be expressed thus,

$$r^2 + Ar\cos\theta + Br\sin\theta + C = 0,$$

which can be immediately deduced from the equation referred to rectangular axes given in Art. 40.

If the centre is on the initial line, then $a = 0$, and the equation becomes

$$r^2 - 2rl\cos\theta + l^2 - \tfrac{1}{4}d^2 = 0.$$

If the origin is on the circumference, then $l = \frac{1}{2}d$, and the equation becomes $r - 2l\cos\theta = 0$.

Some properties of the circle follow at once from this equation. Taking the form

$$r^2 - 2rl\cos\theta + l^2 - \tfrac{1}{4}d^2 = 0,$$

we see that the product of the two values of r is $l^2 - \frac{1}{4}d^2$, which is the same for all values of θ. This is the same as Euclid, Book III., Props. 35 and 36.

Also the sum of the two values of r is $2l\cos\theta$; therefore the coordinates of the middle point of any chord drawn from the origin are $l\cos\theta$ and θ. Hence the polar equation to the locus of that point is $r = l\cos\theta$, which is a circle on l as diameter. This is very obvious geometrically.

N.B.—If θ is greater than $\sin^{-1}\frac{d}{2l}$, the values of r become imaginary. Thus we see that two imaginary quantities have a real sum and a real product.

Exercises on the Circle.

1. Find the coordinates of the centres and length of the radius of the circles whose equations are

$$x^2+y^2-2x-4y-20=0, \quad x^2+y^2+12x-8y+48=0.$$

[*Ans.* (1, 2), and 5; (−6, 4), and 2.

2. Find the points of intersection of the circle $x^2+y^2=65$, and the straight line $3x+y=25$. [*Ans.* (7, 4), and (8, 1).

3. Find the equation of a circle that passes through the origin and makes intercepts a and b on the axes.

[*Ans.* $x^2+y^2-ax-by=0$.

4. Find the equation to the circle that touches both the axes at a distance a from the origin.

[*Ans.* $x^2+y^2-2ax-2ay+a^2=0$.

5. What must be the angle between the axes in order that $x^2-xy+y^2+Ax+By+C=0$ may represent a circle?

[*Ans.* $\frac{2}{3}\pi$.

6. What if $x^2+xy+y^2+Ax+By+C=0$ represent a circle? [*Ans.* $\frac{1}{3}\pi$.

7. If $\frac{x}{a}+\frac{y}{b}=1$ be a tangent to the circle $x^2+y^2=r^2$, find the relation between a, b and c. [*Ans.* $\frac{ab}{\sqrt{a^2+b^2}}=r$.

8. Find the equation to the circle passing through the points (4, 5), (3, 8), (7, 6); and transform it to the centre as origin. [*Ans.* $x^2+y^2-10x-14y+69=0$, $x^2+y^2=5$.

9. Find the centre and radius of the circle

$$r^2-2(\cos\theta+\sqrt{3}\sin\theta)r=5.$$

[*Ans.* $\theta=\frac{1}{3}\pi$, $b=2$, radius $=3$.

10. Given base and vertical angle, find the locus of the vertex. [*Ans.* Let the base $=c$, and vertical angle C, $x^2+y^2-2cy\cot C$.

11. Given base and sum of squares of the two sides of a triangle, find the locus of the vertex.

[*Ans.* $x^2+y^2=\dfrac{2(a^2+b^2)-c^2}{4}$.

12. Given base and ratio of the sides of a triangle, find the locus of the vertex.

[*Ans.* Let the ratio be $m:n$, $x^2+y^2+\dfrac{m^2+n^2}{m^2-n^2}cx+\frac{1}{4}c^2=0$.

N.B. — In the last three examples the base is taken as axis of x, and a perpendicular through its middle point as axis of y.

13. A point moves so that the sum of the squares of its distances from the sides of a square, or from the angles of a square, are constant; shew that in both cases the loci are circles.

14. A point moves so that the sum of the squares of its distances from the sides of a regular polygon, or from the angles of a regular polygon, are constant; shew that these loci are circles.

15. A point moves so that its distance from one angle of a given equilateral triangle is equal to the sum of its distances from the other two; find its locus.

16. If p be the perpendicular from the origin on any tangent to a circle whose polar equation is

$$r^2-2rl\cos(\theta-a)+l^2-\tfrac{1}{4}d^2=0,$$

shew that $p=\dfrac{4(l^2-r^2)-d^2}{4a}$; or, if t be the tangent from the origin, $p=\dfrac{4t^2-d^2}{4d}$.

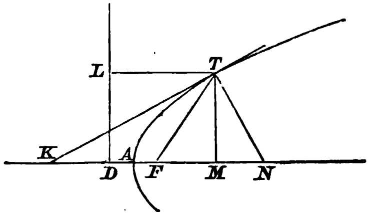
CHAPTER VI.

THE PARABOLA.

54. The curve which comes next to the circle, both in simplicity and in the importance of its applications, is the parabola. It may be defined as

The locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed straight line.

The fixed point (F in figure) is called the *focus* of the parabola, and the fixed straight line (DL) is called the *directrix*. The distance from the focus to the directrix is manifestly bisected by the parabola; for the middle point of the perpendicular through the focus to the directrix is equally distant from the focus and the directrix. This point is called the *vertex* of the parabola; and the distance from the focus to the directrix is generally called $2a$.



The straight line DF through the focus and vertex, which is at right angles to the directrix, is called the *axis* of the parabola.

Any straight line at right angles to the directrix is called a *diameter* of the parabola.

The chord through the focus perpendicular to the axis is called the *Latus Rectum*. This is manifestly equal to $4a$.

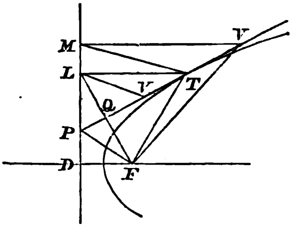
The straight line TN perpendicular to the tangent at the point of contact is called the *normal*. If from any point on a parabola a normal be drawn, and also a perpendicular to the axis, the intercept MN made by them on the axis is called the *subnormal*.

The intercept KM made on the axis between the same perpendicular and the tangent at the point is called the *subtangent.*

The straight line FT joining any point on the curve to the focus is called the *focal distance, focal radius*, or *radius vector.*

The following properties of the parabola will be proved geometrically before proceeding to find its equation:—

55. *The straight line equally inclined to the diameter and focal distance at any point on the parabola is a tangent.*



Let T be a point on the parabola; draw the straight line TQ bisecting the angle FTL between the focal distance (FT) of the point T, and the diameter through it (TL). This straight line TQ has one point on the curve (T), and any other point, however near, may be proved outside the curve. Take any other point on it V; join VL and VF, and draw VM perpendicular to the directrix. Since TQ bisects the vertical angle of the isosceles triangle LTF, it is perpendicular to LF at its middle point. Therefore the triangle LQV is equal in every respect to the triangle FQV; therefore $LV = FV$. But MV is less than LV; therefore MV is less than FV. Consequently the point V is nearer the directrix than the focus, and therefore is outside the parabola. The straight line TQ, as it has one point on the curve and any other point outside the curve, meets the curve but does not cut it, and is therefore a tangent to the parabola.

COR.—From this proposition it can be easily proved that the subnormal is constant and equal to the distance between the focus and directrix. For (Figure, Art. 54) draw the normal TN perpendicular to the tangent at T, and draw TM perpendicular to the axis. Then the angle FTK is equal to the angle LTK, which is equal to the angle FKT. Therefore, since KTN is a right angle, and equal to the sum of the other two in the triangle KTN, it follows that the angle FTN is equal to the angle FNT. Therefore $FN = FT = TL = DM$. Since DM and FN are equal and have FM common, it follows that MN, the subnormal, is equal $FD = 2a$.

56. *If from any point on the curve a perpendicular be let fall on the axis, the square of that perpendicular is equal to twice the product of the intercept from the foot of the perpendicular to the vertex* (A) *of the parabola, and of the distance from the focus to the directrix.*

In the figure for the foregoing proposition, $TM^2 = FT^2 - FM^2$. But $FT = TL = DM$. Therefore

$$TM^2 = DM^2 - FM^2 = (DM + FM)(DM - FM).$$

But $DM + FM = 2AM$, and $DM - FM = FD$.

Consequently

$$TM^2 = 2AM \times FD, \text{ or } = 2AM \times 2a = 4a \,.\, AM.$$

57. *The portion of any tangent between the directrix and point of contact subtends a right angle at the focus.*

*Produce the tangent to meet the directrix at P, and join PF. Since the triangles QLT and QFT have two sides equal ($TL = TF$), and one side PT common, and the angles contained by the equal sides are equal, the triangles are equal in every respect. Therefore the angle PFT is equal to the angle PLT, which is a right angle.

COR.—From this it easily follows that, if tangents be drawn from any point on the directrix, the chord of contact passes through the focus. For the intercept on each tangent from the directrix to the point of contact subtends a right angle at the focus; consequently the straight lines from the focus to the two points of contact make each a right angle with the straight line from the focus to the point on the directrix from which the tangents are drawn; therefore they must form one straight line.

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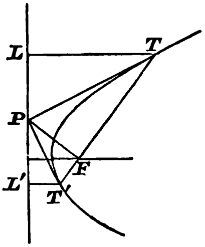
* See figure, Art. 55.

58. *Tangents drawn from any point on the directrix are at right angles.*

Let P be the point on the directrix, PT and PT' the tangents from it; draw FT and FT', also TL and $T'L'$. Then the triangles FPT and LPT are equal (as above), and also similarly the triangles FPT' and $L'PT'$ are equal. Consequently the angles FPT and LPT are equal, also the angles FPT' and $L'PT'$. Therefore the angle TPT' is half the sum of the two right angles at P, and is a right angle.

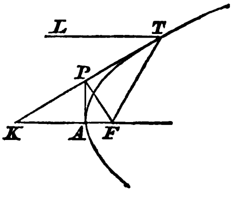


59. *The perpendicular from the focus on the tangent meets it where the tangent at the vertex meets it.*

The perpendicular from the focus FQ (see Figure, Art. 55) is equal to QL, the portion produced to meet the directrix. Also the tangent at the vertex A is parallel to the directrix, for it bisects the angle between the diameter and focal radius at A, which is equal to two right angles; consequently the tangent at A being parallel to DL, and passing through the middle point of FD, it must bisect the opposite side of the triangle FDL, and therefore must pass through Q, the middle point of FL.

Cor.—From this a method of drawing tangents to a parabola from any given external point is easily found. On the straight line joining the given point with the focus and diameter describe a circle. Join the given point with the points where the circle cuts the tangent at the vertex, these straight lines are obviously tangents.

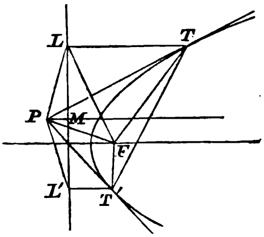
60. *The square of the perpendicular from the focus on the tangent is equal to the product of the distance of the focus to the vertex and the focal distance of the point of contact; or, If p be the length of the perpendicular from the focus on the tangent, and r the focal distance of the point of contact, then $p^2 = ar$.*



The angle $FKP = PTL = PTF$, $\therefore FT = FK$. Also, since FPK is a right-angled triangle, and PA the perpendicular on the hypothenuse, FP is a mean proportional between FA and FK (Euc., Book VI., Prop. 8), or $FP^2 = FA \times FK = FA \times FT$; or $p^2 = ar$.

61. *Tangents from any point* (1) *subtend equal angles at the focus; and* (2) *have equal projections on the directrix; and* (3) *the triangles formed by the tangents and focal distance of the points of contact on each side of the straight line joining the point with the focus are similar.*

(1) Let P be the point, PT and PT' the tangents. Join PF, FT, and FT'; and draw TL, PM, and $T'L'$ perpendicular to the directrix, *i. e.*, diameters of the parabola. The triangles PTF and PTL have the sides FT and LT equal, PT common, and the included angles equal; therefore the triangles are equal in every respect; $PF = PL$, and the angle PFT = the angle PLT.



Similarly, $PF = PL'$, and the angle PFT' = the angle $PL'T'$.

Since PL and PL' are both equal to PF, they are equal to each other; and since, consequently, the angles PLL' and $PL'L$ are equal, and also the angles TLL' and $T'L'L$ are right, the angles PLT and $PL'T'$ are equal; and therefore their equals, the angles PFT and PFT', are equal, which are the angles subtended by the tangents at the focus.

(2) Since $PL = PL'$, and PM is a perpendicular of the isosceles triangle LPL', LL' is bisected in M, therefore $LM = L'M$; but these are the projections of the tangents PT and PT' on the directrix.

(3) Since $PL' = PF$, and $T'F = T'L'$, and TP' is common, the angle $L'PT'$ = the angle FPT'; therefore FPT' is half of

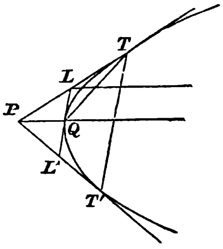
FPL'. Join FL; since PF, PL, and PL' are equal, a circle with its centre at P will pass through L, F, and L'. Therefore the angle $FLL' =$ one-half FPL', and is equal to FPT'. Also the straight line FL is perpendicular to PT (Art. 55); consequently the angles FLL' and PTL are both complements of the angle FLT, and are therefore equal. Therefore $FPT' = FLL' = PTL = PTF$. The angles PFT and PFT' have been proved equal; consequently the triangles PFT and PFT' have two angles in one equal to two angles in the other, and are therefore similar.

62. *If from any external point on a diameter two tangents be drawn, the chord of contact is bisected by the diameter.*

In the preceding figure, join TT', the chord of contact. Then, since TL, PM, and $T'L'$ are all parallel, and M is the middle point of LL', PM, if produced, must pass through the middle point of TT'.

63. *If at the point where the diameter meets the curve a tangent be drawn, it is parallel to the chord of contact of any external point on the diameter.*

Let PT and PT' be the tangents, Q the point where the diameter through P meets the curve, and LL' the tangent to the parabola at the point Q. The diameter through L will (by preceding Article) bisect the chord of contact QT; but as it is parallel to the other diameter PQ, it follows that L must be the middle point of PT. Similarly, L' must be the middle point of PT'.



Therefore LL', joining the middle points of two sides of the triangle PTT', must be parallel to the third side TT'.

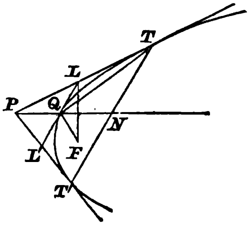
64. *The intercept on the tangent made by two tangents from any point on the diameter is bisected at the point of contact; and the intercept on the diameter between any point on it and the chord of contact of that point is bisected by the curve.*

For, see figure in the next Article, since N is the middle point of TT', and LL' is parallel to TT', $LQ = L'Q$.

Also, as L is the middle point of PT, and LQ is parallel to TN, LQ must bisect PN.

65. *If a tangent and a diameter be drawn at any point on a parabola, any chord parallel to the tangent is bisected by the diameter.*

At T, the extremity of the chord, draw TP, a tangent. If a diameter be drawn at L, where the tangents at Q and T intersect, it will bisect the chord QT. Therefore L must be the middle point of PT, and also PQ must $= QN$. Similarly, the tangent at T' must pass through P, for it must make $PQ = QN$. Therefore TT' is the chord of contact of the point P, and is bisected by the diameter PQ.



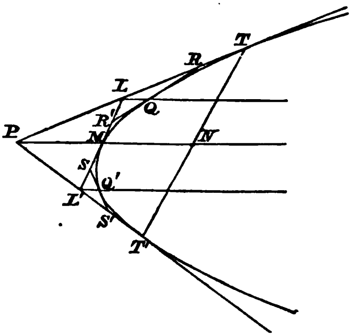
66. *If through any point on a parabola a tangent and a diameter be drawn, and from any other point on the curve a straight line be drawn parallel to the tangent to meet the diameter, the square of that line will be equal to four times the product of the intercept it makes on the diameter from the point of contact and the focal distance of the point of contact.*

Keeping to the construction used in the foregoing figures, it is to be proved that $TN^2 = 4QN \times FQ$.

Since the triangles LFQ and LFT are similar (Art. 60), the angle QLF is equal to the angle LTF, and therefore equal to LTK (TK being the diameter through T), and equal to the angle LPQ. Also, since the tangent QL is equally inclined to the diameter QP and the focal distance FQ, the angles PQL and FQL are equal. The triangles LQP and FQL have two angles in each equal ($LPQ = QLF$ and $PQL = FQL$), and are similar. Therefore $QL^2 = QP \times QF$. But (as in preceding Article), Q being middle point of PN, it follows that $QP = QN$; also that $TN = 2QL$. Therefore $TN^2 = 4QL^2 = 4QP \times QF = 4QN \,.\, QF$.

67. *The area of a segment of a parabola made by any chord is equal to two-thirds of the triangle formed by that chord and the tangents at its extremities.*

Drawing the diameter through the point P at which the tangents meet, and the tangent LL' at the point where it intersects the curve, we have the triangle PML equal to one-half the triangle MNT, because they are on equal bases ($PM = MN$, Art. 63), and the altitude of PML is half the altitude of MNT (as L is the middle point of PT).



Similarly, the triangle $PML' = \frac{1}{2}$ the triangle MNT'. Therefore the triangle $PLL' = \frac{1}{2}$ the triangle MTT'.

If diameters LQ and $L'Q'$ be drawn through the points L and L', and tangents RR' and SS' where these diameters intersect the curve, we have, by the same reasoning, the triangle $LRR' = \frac{1}{2}$ the triangle QTM, and the triangle $L'SS' = \frac{1}{2}$ the triangle $Q'T'M$. Consequently the polygon $RR'SS'P$ is equal to half the polygon $TQMQ'T'$.

If this process of drawing diameters at each point where the tangents meet, and tangents where the diameters intersect the curve, be continued, at each step it is clear that the external polygon so formed is equal to one-half the internal polygon; and when continued long enough the polygons coincide with the curve. Therefore the exterior area made by PT, PT' and the curve is equal to half the area made by the chord TT' and the curve; or, the area cut from the triangle PTT' by the curve is two-thirds of the area of the triangle.

Cor.—*The area of a segment of a parabola made by any chord is equal to two-thirds of the area of the parallelogram formed by the diameters at the extremities of the chord and the tangent parallel to it.*

For draw the diameters at T and T', and produce the tangent LL' to meet them. The parallelogram so formed is equal to the triangle TPT', as they have the same base, and the altitude of the triangle is double that of the parallelogram (PM being $= MN$). Therefore the segment, being two-thirds of the triangle, must be two-thirds of the equal parallelogram.

GEOMETRICAL EXERCISES ON THE PARABOLA.

1. Find the locus of the centres of circles that pass through a given point and touch a given straight line.

2. Given a circle and a diameter of it, find the locus of points equally distant from that diameter and the circumference.

3. The tangent at any point cuts the directrix and the latus rectum produced at points equally distant from the focus.

4. The circle on any focal chord as diameter touches the directrix; and the circle on the focal distance of any point on the parabola touches the tangent at the vertex.

5. The normals at the extremities of any focal chord intersect on the diameter that bisects that chord.

6. If a parabola roll on an equal fixed parabola, commencing with the vertices touching, the focus of the former describes the directrix of the latter.

7. Given three points on a parabola, find its focus and directrix.

8. The diameter of the circle described through the vertex and extremities of the latus rectum is equal to $5a$.

9. If a radius vector be drawn making an angle of 60° with the axis, it will be equal to $4a$; also, with the normal at the point and the axis, it will make an equilateral triangle.

10. If the triangle formed by three tangents to a para-

bola be circumscribed by a circle, the circle will pass through the focus.

11. The chord bisected by any diameter is equal to four times the focal distance of the point where the diameter meets the parabola.

12. If two equal parabolas be described with the same focus, but in opposite directions, the tangents at their points of intersection are at right angles.

13. Find the locus of the middle points of chords drawn from the vertex of a parabola.

[*Ans.* A parabola with the same vertex and axis, the distance from its focus and directrix being half that of the given parabola.

14. The same two points are the focus and vertex of one parabola, and the vertex and focus of another; find the area between the curves.

[*Ans.* $\frac{4}{3}a^2$, where a is the distance between the points.

15. Given two tangents and their chord of contact, to find a series of points on the parabola.

[*Ans.* Join the intersection of the tangents with the middle point of the chord, the middle point of this joining line is a point on the parabola; a straight line through this point parallel to the chord is a tangent to the parabola; with this and the two given tangents two more points can be found; and so on.

16. Given the vertex of a parabola and a chord at right angles to the axis, find a series of points on the curve.

[*Ans.* The method of the preceding example can be easily made to apply to this case.

CHAPTER VII.

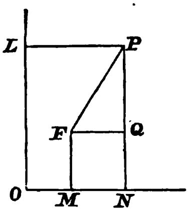
THE EQUATION TO THE PARABOLA.

68. It is plain, from Articles 35 and 36, that the general equation of the second degree can only represent curves which meet a straight line in no more than two points. For if a straight line was intersected in three points, then it is evident, by taking that straight line as one of the axes, that particular values of one must give three corresponding values of the other variable, which must therefore appear in the equation in the third power.

It can be shewn that the general equation of the second degree always represents one of the curves which are formed by the plane sections of a cone. To pròve this would exceed the limits of this book; we will therefore consider these curves as they are represented by their equations when these equations are derived from the simplest properties of the curve. These equations will be found to be of the second degree. Subsequently it will be shewn that the curves are those formed by a plane intersecting a cone.

The simplest case, that of a circle, has been considered as the locus of a point whose distance from a fixed point is constant.

69. The parabola comes next in order of simplicity. And its equation can be derived from the same definition used in the geometrical discussion of its properties; *i.e.*, that it is the locus of a point equally distant from a fixed straight line called the directrix, and a fixed point called the focus.



Taking the directrix as axis of y, and any straight line perpendicular to it as axis of x, the equation to the parabola is easily found. Let the point F, whose coordinates are m and n, be the focus; let P be any point (x, y) on the curve, that is equally distant from F and the directrix OL ($FP=PL$ perpendicular from P to the directrix). Then, drawing the coordinates of P and F and FQ parallel to the axis of x, we have

$$FP^2 = FQ^2+PQ^2 = (ON-OM)^2+(PN-QN)^2.$$

But $FP=PL=ON=x$, $PN=y$, $OM=m$, $FM=n$. Therefore $x^2 = (x-m)^2+(y-n)^2$, or, expanding and collecting terms, $y^2 = 2mx+2ny-(m^2+n^2)$.

The general form of the equation to the parabola, when the directrix is axis of y and any straight line perpendicular to it is axis of x, is $y^2 = Ax+By+C$. In this case the coordinates of the focus are $\frac{1}{2}A$ and $\frac{1}{2}B$, and C must be equal to $-\frac{1}{4}(A^2+B^2)$. Strictly speaking, the equation is

$$y^2 = Ax+By-\tfrac{1}{4}(A^2+B^2).$$

If a straight line parallel to the directrix, at a distance from it k, be taken as axis of y, then for x must be substituted $x \pm k$; this will only affect the indeterminate coefficient. Consequently $y^2 = Ax+By+C$ will represent a parabola, a diameter being axis of x, and a straight line parallel to the directrix being axis of y. The equation to the parabola can be put in a simpler form by taking the perpendicular to the directrix passing through the focus (*i.e.*, the axis of the parabola) as axis of x. To transform the equation $y^2 = 2mx+2ny-(m^2+n^2)$ to the new origin, $y+n$ must be substituted for y, as the ordinate of any point with the old axes exceeds the ordinate to the old by the length n; but the abscissa remains unchanged. Therefore

$$(y+n)^2 = 2mx+2ny-(m^2+n^2),$$

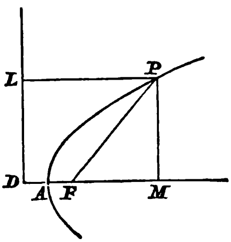
that is, $$y^2 = 2mx-m^2.$$

This is the equation to a parabola when the directrix is axis of y and the perpendicular through the focus is axis of x. This perpendicular is called the *axis* of the parabola, and from the equation the curve is evidently symmetrical on both sides of it. For, in the above equation, for any value of x there are manifestly two equal and opposite values of y.

70. If the vertex which bisects the distance from the focus to the directrix be taken as origin, the equation becomes further simplified. Let the line FD, whose length has been called m, be equal to $2a$, then the equation referred to DF and DM as axes becomes $y^2 = 4ax-4a^2$. To transform the origin to S, the middle point of DF, $x+a$ must be substituted for x (the coordinates of the new origin being $x = a$, $y = 0$). The equation $y^2 = 4ax-4a^2$ becomes $y^2 = 4a(x+a)-4a^2$, that is, $y^2 = 4ax$. This is the simplest form of the equation to the parabola: the origin is on the parabola, at the point where it cuts the principal axis. When the equation of the parabola is mentioned without any specified form being referred to, the form $y^2=4ax$ is to be understood.

If the equation is $y^2 = -4ax$, then the parabola is turned in the opposite direction, the distance of the focus from the vertex being $-a$; that is, the focus is on the negative side of the axis of x. If the equation is $x^2=4ay$, then the focus is on the axis of y and at a distance from the origin equal to a. If it is of the form $x^2=-4ay$, the focus is on the axis of y, at a distance $-a$ on the negative side of the origin.

71. The polar equation may be found when the focus is taken as origin, and the axis as initial line, thus: $FP=r$, and $PFM=\theta$; then, by definition of parabola,



$$FP = PL = DM = DF+FM;$$

that is, $r = 2a+r\cos\theta$; therefore

$$r(1-\cos\theta) = 2a.$$

If θ be measured from the straight line FD, then $FM = r\cos(\pi-\theta)$, and finally

$$r(1+\cos\theta) = 2a.$$

When the vertex is taken as origin, the axis being initial line, the polar equation may be found by substituting $r\sin\theta$ for y, and $r\cos\theta$ for x, then $y^2 = 4ax$ becomes $r^2\sin^2\theta = 4ar\cos\theta$, or $r = \dfrac{4a\cos\theta}{\sin^2\theta}$.

72. The equation of the tangent to a parabola can be found by the same method used to find the equation of a tangent to a circle. Let (x_1, y_1) and (x_2, y_2) be two adjacent points on the parabola $y^2 = 4ax$, then the equation to the secant through them is $y-y_1 = \dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$.

We have now to find a value of $\dfrac{y_2-y_1}{x_2-x_1}$ which will not become indeterminate when $x_2 = x_1$ and $y_2 = y_1$. Since (x_1, y_1) and (x_2, y_2) are both on the parabola, $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$. Therefore, subtracting, $y_2^2 - y_1^2 = 4a(x_2-x_1)$, or $(y_2-y_1)(y_2+y_1) = 4a(x_2-x_1)$. From this it follows that $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{4a}{y_2+y_1}$. The equation to the secant may therefore be written $y-y_1 = \dfrac{4a}{y_2+y_1}(x-x_1)$, which becomes, when the points coincide and the secant becomes a tangent, $y-y_1 = \dfrac{2a}{y_1}(x-x_1)$. Multiplying out, we have

$$yy_1-y_1^2 = 2ax-2ax_1.$$

But $y_1^2 = 4ax_1$; therefore

$$yy_1 = 2ax-2ax_1+4ax_1, \text{ or } yy_1 = 2a(x+x_1).$$

This is the equation of the tangent to a parabola.

The equation to the normal is found by taking the equation of the line passing through (x_1, y_1) and perpendicular to the tangent, that is, $y-y_1 = -\dfrac{y_1}{2a}(x-x_1)$.

The important property of the parabola, that the subnormal at any point is constant and equal to $2a$, easily follows from the equation to the normal. Putting $y = 0$, we get the intercept on the axis made by the normal; thus $-y_1 = \frac{-y_1}{2a}(x-x_1)$, or $2a = x-x_1$, therefore $x = 2a+x_1$; therefore, subtracting the abscissa of the point, we get the subnormal $= 2a+x_1-x_1 = 2a$.

73. *A diameter (i.e., a line at right angles to the directrix) is locus of the middle points of a series of parallel chords.*

This can be easily proved analytically: $y=mx+\beta$, where m is constant and β is variable, represents all straight lines which are parallel to each other. When this is combined with the equation of the parabola $y^2 = 4ax$, or $x = \frac{y^2}{4a}$, by substituting for x, we have $\frac{y-\beta}{m} = \frac{y^2}{4a}$, which gives the ordinates of the extremities of the chords made by the straight lines of the given directrix. Putting this in its simplest form $y^2 - \frac{4ay}{m} + \frac{4a\beta}{m} = 0$, we find that the sum of the values of y is $\frac{4a}{m}$ (for it is equal to the coefficient of the second term with its sign changed), therefore the ordinate of the middle point which is equal to half this sum is $\frac{2a}{m}$. This is independent of β; therefore the middle point of the chord made by $y = mx+\beta$ must always lie on the line $y = \frac{2a}{m}$; *i.e.*, on a line parallel to the axis of x. (From this property the straight lines perpendicular to the directrix are called diameters.)

N.B.—It depends on the value of β whether the straight line cuts the parabola or not. If it does not, then the roots of y in $y^2 - \frac{4a}{m}y + \frac{4a\beta}{m} = 0$ are imaginary; but their sum is real. Compare Art. 53.

74. *To find the equation to the parabola referred to any diameter, and the tangent where it meets the curve, as axes.*

In Article 65 there is a geometrical solution of this, $TN^2 = 4\,QN\,.\,QF$ is the same as $y^2 = 4a'x$, where a' is the focal distance of the new origin; $QF = a'$, $TN = y$, and $QN = x$ referred to QN and QL as axes.

But it may be as well to do this analytically. Let the coordinates of the new origin be h, k. Transforming the equation to that origin without changing the direction of the axes, the equation $y^2 = 4ax$ becomes

$$(y+k)^2 = 4a\,(x+h)$$

(Art. 31). Changing the direction of the axis of y, and taking θ as the angle between the diameter and tangent, which becomes axis of y, we have in the formula given in Article 32, since $w = 90°$, $\alpha = 0°$, and $\beta = \theta$, to put for x, $x+y\cos\theta$, and for y to put $y\sin\theta$. The above equation becomes $(y\sin\theta+k)^2 = 4a\,(x+y\cos\theta+h)$,

or, expanding,

$$y^2\sin^2\theta+2y\,(k\sin\theta-2a\cos\theta)+k^2-4ah = 4ax.$$

But since (h, k) is a point on the parabola $k^2 = 4ah$; also drawing the normal at that point, we have evidently, from the subnormal being equal to $2a$,

$$k = 2a\cot\theta, \quad \text{or} \quad k\sin\theta = 2a\cos\theta.$$

The coefficient of y becomes 0; therefore the equation becomes $y^2\sin^2\theta = 4ax$, or $y^2 = 4\dfrac{a}{\sin^2\theta}x$.

It can be further shewn that a', the focal distance of the new origin, is equal to $\dfrac{a}{\sin^2\theta}$.

For its focal distance, being equal to its distance from the directrix, is equal to $a+h$. Consequently, we have

$$a' = a+h = a+\frac{k^2}{4a} = a+a\cot^2\theta$$

$$= a\,(1+\cot^2\theta) = a\operatorname{cosec}^2\theta = \frac{a}{\sin^2\theta}.$$

The equation $y^2 = 4\frac{a}{\sin^2\theta}x$ may be written $y^2 = 4a'x$, where a' is the focal distance of the origin.

75. The properties of the chord of contact, and of poles and polars, proved for the circle, are applicable to the parabola, and to all the conic sections.

To find the equation of the chord of contact of two tangents from a point (h, k) *to a parabola* $y^2 = 4ax$.

Let x_1, y_1 be the coordinates of the point of contact of one tangent, and x_2, y_2 of the point of contact of the other. The equation of the first tangent is $yy_1 = 2a(x+x_1)$; since this passes through (h, k), we have $ky_1 = 2a(h+x_1)$. Again, the equation to the other tangent is $yy_2 = 2a(x+x_2)$, and since this also passes through (h, k), $ky_2 = 2a(h+x_2)$. Hence it follows that $ky = 2a(x+h)$ is the equation to the chord of contact. For it is obviously the equation to a straight line, and it is satisfied by x_1, y_1 and x_2, y_2, the coordinates of the two points of contact, therefore it is the straight line joining them.

76. *If chords are drawn to a parabola through a given point, and tangents drawn at the extremities of each chord, the locus of their intersections is a straight line.*

Let (h, k) be the coordinates of the point through which the chords are drawn, and let the coordinates of the point at which the tangents at the extremities of one of the chords meet be x_1, y_1.

Then the equation to the corresponding chord of contact is $yy_1 = 2a(x+x_1)$. But this chord passes through the point (h, k); therefore $ky_1 = 2a(k+x_1)$. Similarly it can be proved that the coordinates of the points of contact of two tangents drawn at the extremities of any other chord passing through the point (h, k) satisfy the equation

$$ky = 2a(x+h).$$

This equation, which represents a straight line, represents the required locus.

77. *If from any point in a straight line tangents be drawn to a parabola, the chords of contact will pass through a given point.*

Let $Ax+By+C=0$ be the equation to the straight line, and h, k the coordinates of a point on it. The equation to the corresponding chord of contact is $ky=2a(x+h)$. But, since (h, k) is a point on the given straight line,

$$Ah+Bk+C=0, \text{ therefore } k=-\frac{Ah+C}{B}.$$

The equation to the chord of contact may therefore be written

$$-\frac{Ah+C}{B}y=2a(x+h), \quad \text{or} \quad -h\left(\frac{Ay}{B}+2a\right)=2ax+\frac{C}{B}y.$$

Now, whatever be the value of h, the straight line represented must pass through the intersection of the two straight lines $\frac{A}{B}y+2a=0$ and $2ax+\frac{C}{B}y=0$, that is, through the point $x=\frac{C}{A}$, $y=-\frac{2aB}{A}$.

The general observations on the properties of the chord of contact and poles and polars, made in reference to the circle, are applicable also to the parabola, and indeed to the conic sections generally; and may be proved by the same method employed in these cases.

It will form an easy and useful exercise for the student to prove analytically the properties of which geometrical proofs have been given in Articles 54 to 66.

78. *To find the envelope of the straight line* $y=mx+\frac{a}{m}$ *(m being variable).*

By the *envelope* of a straight line is meant a curve which under a given condition it must touch. This condition may be given analytically or geometrically.

In Art. 42 it was proved that any straight line whose equation was of the form

$$y=mx+c\sqrt{1+m^2},$$

must touch a circle with its centre at the origin, and its radius equal to c; no matter what the value of m. This might be expressed by saying that the circle $x^2+y^2=c^2$ was the *envelope* of the straight line $y=mx+c\sqrt{1+m^2}$.

Again, if a chord of given length be inscribed in a circle, it must touch a concentric circle, the square of whose radius is equal to the difference of the squares of the radius of the given circle and of half the given length of chord. This concentric circle is the *envelope* of chords of the given length in the given circle, or of all lines making chords of the given length in the given circle.

We will now proceed to shew that the straight line $y=mx+\frac{a}{m}$, whatever be the value of m, must touch the parabola $y^2=4ax$. See figure to Art. 59.

Take, on the axis of y, a length AP equal to the indeterminate coefficient of the straight line, that is, its intercept on the axis of y. Take, on the axis of x, $AF=a$. Join FP. Since $\tan APF=\frac{AF}{AP}=\frac{a}{\frac{a}{m}}=m$, therefore the angle APF = the angle PKA which the straight line makes with the axis of x. Consequently, the triangles APF and PKF, having one angle common and another in each equal, must be equal in every respect, therefore the angle FPK is right, therefore the straight line KP must be a tangent to the parabola $y^2=4ax$, for it fulfils the condition that the perpendicular from the focus meets it on the axis of y, which is the tangent at the vertex.

EXERCISES ON THE PARABOLA.

1. Find the equations to the tangent and normal at the extremity of the Latus Rectum

[*Ans.* $y-x=a$, and $y+x=3a$.

2. Trace the curves (*i.e.*, find coordinates of focus and

position of directrix) $y^2 = 5x$, $y^2 = 6x+10$, $x^2 = 3y-2$, $y^2 = 4x+12y-10$, $2x^2 = 18x+6y+27$.

[*Ans. to last.* Coordinates of focus are $\frac{9}{2}$, $\frac{3}{2}$; directrix $4y = 33$.

3. Find the equation to the circle that passes through the vertex and the extremities of the Latus Rectum of the parabola $y^2 = 4ax$. [*Ans.* $x^2+y^2-5ax = 0$.

4. Find the equations to the tangent and normal at the extremities of the Latus Rectum.

[*Ans.* $y-x = a$, and $y+x = 3a$.

5. Find the length of the focal chord of a parabola inclined at an angle θ to the axis. [*Ans.* $\frac{4a}{\sin^2\theta}$.

6. Find the area of the triangle formed by this chord, and the tangents at its extremities. [*Ans.* $\frac{4a^2}{\sin^3\theta}$.

7. A normal is drawn at the extremity of the Latus Rectum, find the coordinates of the point where it meets the parabola again, and the length of the chord it makes.

[*Ans.* $x = 9a$, $y = -6a$, length of chord $= 2^{\frac{7}{2}} a$.

8. In the parabola $y^2 = Ax+By+C$, what straight lines are represented by the equations

$$y = 0 \quad \text{and} \quad Ax+By+C = 0?$$

[*Ans.* A diameter, and the tangent where that diameter meets the curve.

9. If from any point on a parabola a tangent and a chord be drawn, the intercept on any diameter between the chord and tangent is cut by the curve in the same ratio as the diameter cuts the chord.

10. From the intersection of the axis and directrix chords are drawn, find the locus of their middle points.

[*Ans.* The parabola $y^2 = 2a(x+a)$.

11. Find the locus of the intersection of the normals at the extremities of a focal chord.

[*Ans.* The parabola $y^2 = ax-3a^2$.

12. The harmonic mean between the segments of a focal chord $= 2a$.

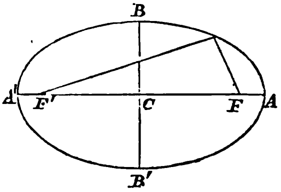
CHAPTER VIII.

THE ELLIPSE.

79. The next curve to be considered is the Ellipse. This is of the form commonly called oval, and can be very easily described in the following manner:—If the ends of a string be fastened at two points, a pencil which keeps the string extended will, as it moves, describe an ellipse. The two fixed points are called the *foci*. The ellipse may therefore be defined as the locus of the vertex of a triangle of which the base and the sum of the sides are given, the extremities of the base being the foci. This property of the curve may be thus expressed:—The sum of the focal distances (or radii) of any point on it is constant. The form of the curve will depend on the ratio of the distance between the foci to the sum of the focal distances; that is, the ratio of the distance between the two fixed points to which the string is fastened and of the length of the string. Thus if, while the length of the string or sum of the focal radii remains the same, the foci are moved nearer each other, the ellipse will become more like a circle; and when they coincide, it will become a circle with a radius equal to half the sum of the focal radii. If, on the other hand, the foci be moved farther apart, the ellipse will become a more flattened oval. If the sum of the focal radii be called $2a$, and the distance between the foci be called $2c$, then this ratio, on which the form of the ellipse depends, is $\frac{c}{a}$. This is called the *excentricity* of the ellipse, and is usually denoted by the letter e. This must always be fractional, as the length of the string manifestly cannot be less than the distance between its extremities. It is in proportion to the

value of this fraction that the ellipse differs from a circle. In the circle it becomes = 0, the distance between the foci being nothing. It must be carefully borne in mind that it is not the absolute magnitude of the distance between the foci that determines the form of the ellipse, but its proportion to the length of the describing string or sum of the focal radii. For example, the distance between the foci of the elliptical orbit of the planet Saturn is 90,000 miles, yet if it were drawn to scale so as to be seen by us, its difference from a circle would be hardly perceptible.

It is evident that, when the describing pencil passes the straight line joining the foci (which is called the *major axis* of the ellipse), it cuts it at points A and A' (called the *vertices*), so that $AA' = 2a$. For



$$FA + F'A = 2a,$$

and $$FA' + F'A' = 2a.$$

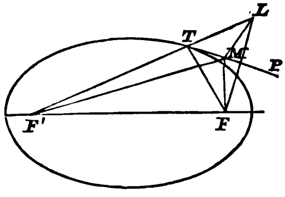
Therefore $$FA = F'A',$$

and $$AA' = F'A' + F'A = FA + F'A = 2a.$$

Also, when the describing pencil passes a straight line perpendicular to FF' at its middle point C (this point is called the *centre* of the ellipse), it is at its shortest distance from that point, and the distances from the foci are equal, and are each $= a$. This distance from the centre is called b, and is manifestly equal to $\sqrt{a^2 - c^2}$. The intercept (BB') on this straight line made by the ellipse is called the *minor axis*, and is $= 2b$. Some of the leading properties of the ellipse can be very easily proved geometrically, as in the case of the parabola.

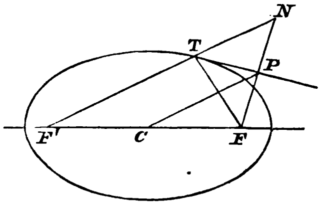
80. *The tangent at any point on the ellipse is the straight line equally inclined to the focal radii of that point.*

Let T be any point on an ellipse of which F and F' are the foci. Draw TM equally inclined to the focal radii (*i.e.*, bisecting the external angle of the triangle FTF'), and it is a tangent to the ellipse. For it has one point T on the



ellipse, and any other point on it, as M, may be proved to be outside the ellipse. Draw FM and $F'M$; produce $F'T$ till $TL = FT$; join ML and $F'M$; Since $F'L = F'T + FT$, $F'L = 2a$. But $F'M + ML > F'L$; and ML manifestly $= FM$, the triangles MTL and MTF being equal in every respect. (Euc., Book I., Prop. 4.) Consequently $FM + F'M > 2a$; therefore the point M must be outside the ellipse, and the straight line TP, having one point on the ellipse and every other point outside the ellipse, is a tangent. (N.B.—It is manifest that for any point outside the ellipse the sum of the focal distances exceeds $2a$, and for any point inside the ellipse the sum of the focal distances is less than $2a$.)

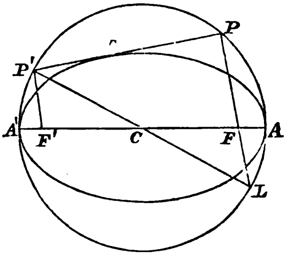
81. *To find the locus of the foot of the perpendicular from the focus on the tangent.*



Let P be the point where the perpendicular from the focus F, on any tangent TP, meets it. (This point is called the *foot* of that perpendicular.) Produce FP to meet $F'T$ produced in N. Then, by Euc., Bk. I., Prop. 26, the triangles PTF and PTN are equal in every respect; P is the middle point of FN, and $TN = TF$; consequently $F'N = F'T + FT = 2a$. Join C, the centre of the ellipse, with P; then $CP = \frac{1}{2}F'N$, since it joins the middle points of the sides of the triangle $F'FN$; consequently $CP = a$. Therefore, the foot of the perpendicular from the focus on any tangent being at a distance from the centre equal to a, the locus of it is a circle whose centre is the centre of the ellipse, and whose radius is equal to the semi-axis major.

82. *The product of the perpendiculars from the foci on the tangent equals the square of the semi-axis minor* ($=b^2$).

This is easily proved by the auxiliary circle, as the perpendiculars FP and $F'P'$ meet the tangent on its circumference, by preceding Article. Produce $P'C$ to meet PF in L; this point must also be on the auxiliary circle, as the angle $P'PF$ is right. The triangles FCL and $F'CP'$ are manifestly equal (Euc., Bk. I., Prop. 4); therefore $FL=F'P'$. Therefore



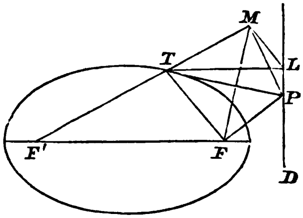
$$FP\times F'P' = FP\times FL = AF\times FA'$$

(Euc., Bk. III., Prop. 35)

$$= (a-c)\,(a+c) = a^2-c^2 = b^2.$$

83. *If a perpendicular be erected at the focus to the radius vector of any point, and produced to meet the tangent to the ellipse at that point, the locus of their intersection will be a straight line.*

Let T be any point on the ellipse, and TP the tangent at it, P being its point of intersection with FP, the perpendicular to FT at F. Produce $F'T$, and make $TM=TF$; and join PM. The triangles PTF and PTM are manifestly equal in every respect (Euc., Bk. I., Prop. 4); consequently $PM=PF$, and the angle $PMT=$ the angle PFT, and is a right angle. Therefore



$$F'P^2-PM^2\ (=F'P^2-FP^2) = F'M^2$$
$$= (F'T+FT)^2 = 4a^2,$$

Since the difference of the squares of the distances of the point P from the points F and F' is constant, the locus of P must be a straight line perpendicular to the line joining FF'. (Euc., Book II., Prop. 13, Cor.)

This straight line is called the *directrix*, and it has a relation to the ellipse similar to the relation of the directrix to the parabola. In the ellipse there are manifestly two directrices corresponding to the two foci.

84. *The ratio of the distances of any point on the ellipse from the focus and directrix is constant* ($= e$).

Let the same construction remain; and draw TL perpendicular to the directrix DP, and join MF and ML. Because the angles TFP, TMP, and TLP are right, a circle on TP as diameter will pass through F, M, and L. Consequently the angle TLM is equal to the angle TFM. But the angle TFM is equal to the angle TMF (for $TM = TF$). Therefore the angle TLM is equal to the angle FMF'. Also, since TL is parallel to FF', both being at right angles to the directrix, the angle MTL is equal to the angle $MF'F$. Consequently the triangles MTL and $MF'F$ are similar. Therefore $\frac{TM}{TL} = \frac{FF'}{F'M} = \frac{2c}{2a}$. But $TM = TF$. Therefore $\frac{TF}{TL} = \frac{c}{a} = e$.

85. *The external angle between the focal radii of any two points on the ellipse is bisected by the straight line joining the focus with the intersection of the directrix and the chord joining the points.*[*]

For if T and T' be the points, and D the point where the chord joining them intersects the directrix, it is evident, by drawing the perpendiculars to the directrix, that

$$DT : DT' = TL : T'L' = FT : FT'.$$

The straight line FD cuts the base produced of the triangle TFT' in the ratio of the sides, and therefore is the bisector of the external vertical angle.

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[*] The student can easily supply the figure.

86. *To find the relation between the radius vector of a point and the perpendicular from the focus on the tangent at that point.*

It is evident, from the figure in Article 82, that the triangles TFP and $TF'P$ are similar. Therefore

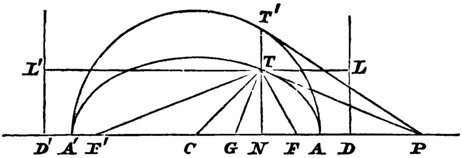
$$FP : F'P' = FT : F'T, \quad \text{or} \quad p : p' = r : r'.$$

But, by the same Article, $p \times p' = b^2$. Therefore

$$p : \frac{b^2}{p} = r : r', \quad \text{or} \quad \frac{p^2}{b^2} = \frac{r}{r'} = \frac{r}{2a-r}, \quad \text{or} \quad p^2 = \frac{b^2 r}{2a-r}.$$

87. *If P be the intersection of the major axis with any tangent, and N the foot of the perpendicular from the point of contact on the major axis, C being the centre of the curve, then $CN \times CP = a^2$.*

Since TP is the bisector of the external vertical angle of



the triangle FTF', it cuts the base produced in the ratio of the sides. Consequently

$$\frac{FP}{FP} = \frac{FT}{F'T} = \frac{TL}{TL'} = \frac{ND}{ND'}.$$

Compounding the ratios by addition and subtraction, we have

$$\frac{F'P+FP}{F'P-FP} = \frac{ND'+ND}{ND'-ND},$$

But $FP+F'P = 2CP$, and $F'P-FP = 2CF = 2c$.

Also $ND'+ND = 2CD = 2\frac{a}{e}$, and $ND'-ND = 2CN$.

Therefore $$\frac{2CP}{2c} = \frac{2\frac{a}{e}}{2CN},$$

or $$CP \times CN = c \times \frac{a}{e} = ea \times \frac{a}{e} = a^2.$$

88. *If G be the point where the normal meets the major axis, then* $NG : CN = b^2 : a^2$.

The normal being at right angles to the bisector of the external vertical angle of FTF', bisects the vertical angle.

Consequently $$\frac{FG}{F'G} = \frac{FT}{F'T} = \frac{ND}{ND'}.$$

From this it follows that

$$\frac{F'G-FG}{F'G+FG} = \frac{ND'-ND}{ND'+ND'}, \text{ or } \frac{2CG}{2CF} = \frac{2CN}{2CD'}, \text{ or } \frac{CG}{c} = \frac{CN}{a \div e}.$$

From this, $$\frac{CG}{CN} = \frac{c}{a \div \frac{c}{a}} = \frac{c^2}{a^2}.$$

Therefore $$\frac{CN-CG}{CN} = \frac{a^2-c^2}{a^2}, \text{ that is, } \frac{NG}{CN} = \frac{b^2}{a^2}.$$

89. *If TN, the perpendicular on the major axis, be produced to meet the auxiliary circle in T′, then* $T'N : TN = a : b$.

Since $CP \times CN = a^2$, *i.e.*, $CP \times CN = CT'^2$ (CT' being the radius of the auxiliary circle); therefore $CT'P$ is a triangle right-angled at T'. Also, by the preceding Article, $\frac{NG}{CN} = \frac{b^2}{a^2}$. Multiplying the numerator and denominator of the first fraction by PN, we have $\frac{PN \times NG}{PN \times CN} = \frac{b^2}{a^2}$. But since GTP is a right-angled triangle, and TN a perpendicular on the hypothenuse, $PN \times NG = TN^2$; and since $CT'P$ is a triangle right-angled at T', $PN \times CN = T'N^2$.

Therefore $$\frac{TN^2}{T'N^2} = \frac{b^2}{a^2}.$$

COR. $TN^2 : AN \times NA' = b^2 : a^2$.

For, since $AT'A'$ is the angle in a semicircle, it is a right angle; therefore $AN \times NA' = T'N^2$. Consequently

$$\frac{TN^2}{AN \times NA'} = \frac{b^2}{a^2}.$$

90. *The area of an ellipse* $= \pi ab$.

From this the area of the ellipse can be easily deduced. If the semi-ellipse and corresponding semicircle of the auxiliary circle be divided into a number of infinitely thin slices by straight lines parallel to the minor axis, each slice of the semi-ellipse is to the corresponding slice of the semicircle as b to a. Consequently their sums (*i. e.*, the semi-ellipse and semicircle) are as b to a.

The area of the ellipse is therefore to the circle on the major axis as b to a; and, the area of the circle being πa^2, the area of the ellipse is πab. Consequently the area of an ellipse is equal to the area of a circle whose radius is a mean proportional between the major and minor semi-axes of the ellipse.

GEOMETRICAL EXERCISES ON THE ELLIPSE.

1. If the extremities of a focal chord be joined with the point where the major axis meets the directrix, the angle between the joining lines is bisected by the major axis.

2. Shew that the least angle between the tangent and focal radius at a point is at the extremity of the minor axis.

3. Draw a pair of tangents from a given external point.

4. Draw a normal from a given point on the minor axis.

5. The Latus Rectum and the major and minor axes are in continued proportion.

6. A pair of tangents from any point subtend equal angles at each of the foci.

7. If a circle be described touching the major axis at a focus and passing through the extremity of the minor axis, then the semi-axes minor and major and the diameter of the circle will be in continued proportion.

8. Prove that the circle on any focal radius as diameter touches the auxiliary circle.

9. Given one focus and three points on the ellipse, describe it.

10. Given the foci and a tangent to the ellipse, describe it.

11. If the tangent at any point T on the ellipse meets the minor axis produced in K, and TM be drawn perpendicular to the minor axis, then (C being the centre) $CK \times CM = b^2$.

CHAPTER IX.

EQUATION TO THE ELLIPSE.

91. From the method of describing the ellipse by a string whose ends are fastened at the foci, in other words, from the property that the sum of the focal radii is constant $(r+r'=2a)$, the equation of the ellipse can be very easily found. Take the centre as origin, the major axis as axis of x, and the minor axis as axis of y.

Let r and r' be the focal distances of any point on the ellipse, its coordinates being x and y. Then

$$r=\sqrt{y^2+(c+x)^2}, \text{ and } r'=\sqrt{y^2+(c-x)^2}.$$

Consequently,

$$2a \text{ (by hypothesis } = r+r')$$
$$=\sqrt{y^2+(c+x)^2}+\sqrt{y^2+(c-x)^2}.$$

Squaring both sides, we have

$$4a^2=2y^2+(c+x)^2+(c-x)^2+2\sqrt{[y^2+(c+x)^2][y^2+(c-x)^2]}.$$

Bring over to the left hand all that is not under the radical sign, and substituting for c^2 its value a^2-b^2, we have

$$a^2+b^2-x^2-y^2=\sqrt{[y^2+(c+x)^2]\,[y^2+(c-x)^2]}.$$

Again, squaring both sides, substituting for c^2 its value a^2-b^2, and cancelling, we finally obtain $b^2x^2+a^2y^2=a^2b^2$; which may be written $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. This is the simplest form of the equation to the ellipse.

92. *The focal distances r and r' of any point on the ellipse are respectively $a+ex$, and $a-ex$.*

For $r^2=y^2+(c+x)^2$ and $r'^2=y^2+(c-x)^2$. Therefore $r^2-r'^2=4cx$; therefore

$$(r+r')(r-r')=4cx, \text{ or } 2a\,(r-r')=4cx.$$

Therefore the difference of the radii vectores

$$(r-r') = 2\frac{cx}{a} \text{ or } 2ex,$$

(Art. 79). Since the sum of the focal distances is $2a$, and their difference is $2ex$, they must respectively be

$$a+ex \text{ and } a-ex.$$

EQUATIONS TO THE TANGENT AND NORMAL.

93. These can be found by the same method as used for the circle and parabola. The equation to the secant through two points on the ellipse (x', y'), and (x'', y''), is

$$y-y' = \frac{y''-y'}{x''-x'}(x-x').$$

Putting in the condition that both points are on the ellipse and satisfy its equation, we have

$$a^2y'^2+b^2x'^2 = a^2b^2, \text{ and } a^2y''^2+b^2x''^2 = a^2b^2;$$

and, subtracting, we obtain

$$a^2(y''^2-y'^2)+b^2(x''^2-x'^2) = 0.$$

From this it follows that

$$\frac{y''-y'}{x''-x'} = -\frac{b^2}{a^2}\frac{x''+x'}{y''+y'}.$$

Therefore, when the two points coincide, and the secant becomes a tangent, the equation becomes

$$y-y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x-x').$$

This may be written

$$a^2y'y+b^2x'x = a^2y'^2+b^2x'^2 = a^2b^2,$$

since the point (x', y') is on the ellipse. The equation to the tangent may therefore be written $\frac{xx'}{a^2}+\frac{yy'}{b^2} = 1$, which is its simplest form.

94. *Equation to the normal.*—The equation to the tangent may be written $y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}$.

The normal being at right angles to it and passing through the point (x', y'), its equation is evidently

$$y - y' = \frac{a^2y'}{b^2x'}(x - x').$$

95. *The equation to the auxiliary circle having its centre at the origin and its radius* $=a$, *is manifestly* $x^2 + y^2 = a^2$.

From this the property proved in Article 89 follows immediately, that if the ordinate of any point on the ellipse be produced to meet the auxiliary circle, the ordinate of the corresponding point on the circle is to the ordinate of the point on the ellipse as a to b. For the abscissæ of both points is the same, and the equation to the ellipse may be written $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, and the equation to the circle $y^2 = a^2 - x^2$; when x is the same in both, then it is that the y in the first is to the y in the second as b to a.

96. The polar equation to the ellipse is easily found when the centre is origin and major axis the initial line, by substituting $r\cos\theta$ for x, and $r\sin\theta$ for y in the equation in Art. 91, and it becomes

$$r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2, \text{ or } \frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2} = \frac{1}{r^2}.$$

When the focus is taken as origin, we have $r = eTL$,* that is, $r = e(FD + FN) = e(p + r\cos\theta)$.

To find the value of p in terms of a, we have

$$p = FD = FA + AD = FA + \frac{FA}{e},$$

that is, $= a - c + \frac{a-c}{e} = a - ea + \frac{a-ea}{e} = \frac{a}{e} - ea = \frac{a(1-e^2)}{e}$.

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* See figure, Art. 87.

Putting in this value for p, we have

$$r = a(1-e^2) + er\cos\theta, \quad \text{or} \quad r = \frac{a(1-e^2)}{1-e\cos\theta}.$$

97. *If p be the length of the perpendicular from the centre on the tangent, and α its inclination to the major axis, $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.*

From Arts. 12 and 92,

$$x\cos\alpha + y\sin\alpha = p, \quad \text{and} \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

must both represent the tangent. Therefore, the coefficients being proportional, it follows that

$$\frac{\cos\alpha}{p} = \frac{x'}{a^2}, \quad \text{and} \quad \frac{\sin\alpha}{p} = \frac{y'}{b^2}.$$

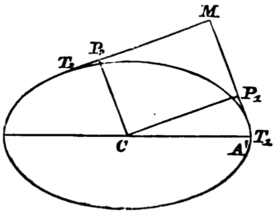
Therefore $\dfrac{a\cos\alpha}{p} = \dfrac{x'}{a}$, and $\dfrac{b\sin\alpha}{p} = \dfrac{y'}{b}$.

Squaring and adding, we have

$$\frac{a^2\cos^2\alpha}{p^2} + \frac{b^2\sin^2\alpha}{p^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1;$$

[since (x', y'), the point of contact of the tangent, is on the ellipse], or $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

98. *To find the locus of intersection of two tangents that are at right angles to each other.*



Let MT_1 and MT_2 be two tangents at right angles, and let the angle ACP_1 be α, and the angle $ACP_2 = \alpha + 90°$. Then

$$CP_1^2 = a^2\cos^2\alpha + b^2\sin^2\alpha,$$

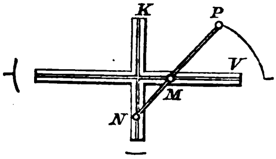
also $CP_2^2 = a^2\sin^2\alpha + b^2\cos^2\alpha$, adding

$$CP_1^2 + CP_2^2 = a^2 + b^2 = (P_1P_2)^2.$$

But since CP_1MP_2 is a rectangle, $CM = P_1P_2$. Consequently

M is at a distance from $C = \sqrt{a^2+b^2}$, and its locus is a circle with its centre at C and radius equal to $\sqrt{a^2+b^2}$.

99. *An instrument is frequently used for drawing the ellipse called the Trammel.*



It consists of a straight line which as it moves has two points on it constrained to keep on two fixed straight lines at right angles to each other. Any point on the moving straight line will describe an ellipse; and its equation can be at once deduced.

Let P be any point with coordinates x, y on the moving line, the fixed lines being taken as axes. PM will evidently be the minor axis b of the ellipse, and PN the major axis a, $PV=y$, and $PK=x$. Now

$$\frac{PK}{PN} = \sin PNK = \cos PMV; \text{ and } \frac{PV}{PM} = \sin PMV;$$

therefore $\dfrac{PK^2}{PN^2}+\dfrac{PV^2}{PM^2} = \cos^2 PMV+\sin^2 PMV = 1.$

Putting in the values, we get $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The Trammel is also called the Elliptic Compass; it is made so that the positions of M, N, or the tracing point P, can be changed, in order that ellipses of different areas or of different excentricities can be described.

CONJUGATE DIAMETERS.

100. If (x', y') be a point on an ellipse, the equation to the diameter through it is $x'y = y'x$ ....(1),

as the centre is also the origin. The equation to the diameter parallel to the tangent at that point is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=0 \quad \text{........................(2).}$$

For it is the equation to a straight line passing through the centre and origin, and it is parallel to the straight line $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$, which is the equation to the tangent. (Art. 92.)

These diameters, *i.e.*, that through any point on an ellipse and that parallel to the tangent at that point, are called a pair of *conjugate diameters*, and have several important relations. The angles they make with the axis of x are thus connected. Let θ be the angle made by (1), and θ' the angle made by (2). Now (1) may be written $y = \frac{y'}{x'}x$, and (2) may be written $y = \frac{-x'}{y'}\frac{b^2}{a^2}x$; consequently (Art. 10), $\tan\theta = \frac{y'}{x'}$, and $\tan\theta' = \frac{-x'}{y'}\frac{b^2}{a^2}$. Therefore $\tan\theta \,.\, \tan\theta' = \frac{b^2}{a^2}$.

101. *Conjugate diameters bisect the chords parallel to each other.*

$x'y = y'x + C$ will represent, by giving different values to C, every chord parallel to the diameter passing through the point (x', y'), and it can be written $y = \frac{y'x + C}{x'}$. Putting in this value of y in the equation of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2,$$

the resulting quadratic in x will give the abscissæ of the extremities of the chord; and the abscissa of the middle point of the chord will be half the sum of the abscissæ of the extremities; that is, half the sum of the roots of the equation

$$b^2x^2 + \frac{a^2(y'x + C)^2}{x'^2} = a^2b^2.$$

Now the sum of the roots is equal to the coefficient of the second term with its sign changed, when the equation is put into its simplest form. Reducing this equation to find the coefficient of the second term, we may neglect the terms which do not involve x, and we have

$$b^2x^2+\frac{a^2y'^2}{x'^2}x^2+\frac{2a^2y'C}{x'^2}x, \text{ or } \frac{b^2x'^2+a^2y'^2}{x'^2}x^2+\frac{2a^2y'C}{x'^2}$$

and since (x', y') is a point on the ellipse,

$$b^2x'^2+a^2y'^2=a^2b^2;$$

this becomes $\frac{a^2b^2}{x'^2}x^2+\frac{2a^2y'C}{x'^2}x$ which gives as coefficient of x, $\frac{2y'C}{b'^2}$; and the abscissa of the middle point of the chord is $\frac{-a^2y'C}{b^2}$.

In the same way it will be found that the ordinate of the middle point of the chord is $\frac{x'C}{a^2}$.*

But these values of x and y satisfy the equation of the conjugate diameter $\frac{xx'}{a^2}+\frac{yy'}{b^2}=0$. Consequently, the middle point of a chord parallel to the diameter $x'y = y'x$ is on the conjugate diameter.

The same process may be pursued with chords parallel to the other diameter $\frac{xx'}{a^2}+\frac{yy'}{b^2}=0$.

These must all be represented by

$$x=-\frac{a^2}{x'}\left(\frac{yy'}{b^2}+C\right).$$

Putting this value of x in the equation to the ellipse, we have

$$\frac{b^2a^4}{x'^2}\left(\frac{yy'}{b^2}+C\right)^2+a^2y^2=a^2b^2.$$

Proceeding to obtain the coefficient of y in this equation in its simplest form, it becomes

$$\left(\frac{a^4y'^2}{b^2x'^2}+a^2\right)y^2+2C\frac{a^4y'}{x'^2}y.$$

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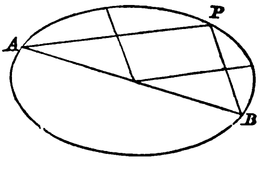
* In the first step, C must be taken with opposite sign, as is obviously the case.

This becomes $\frac{a^4b^2}{b^2x'^2}y^2+2C\frac{a^4y'}{x'^2}$; and the coefficient of y is $2Cy'$. Therefore the y of the middle point $=-Cy'$. Similarly the x of the middle point is found to be $-Cx'$; and these values manifestly satisfy the equation $x'y = y'x$.*

102. *Conjugate diameters are parallel to supplemental chords. (Supplemental chords are those drawn from any point on the curve to extremities of any diameter.)*



Let P be the point, and A and B the extremities of a diameter. The diameter through the middle point of PB is parallel to AP (as it joins the middle points of the sides of the triangle ABP); also the diameter through the middle point of AP is parallel to BP. These diameters therefore bisect the chords parallel to each other, and are conjugate.

103. *To find the coordinates of the extremity of the diameter conjugate to a diameter passing through a given point.*

Let x', y' be the coordinates of the given point, the equation of the diameter conjugate to that passing through it is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=0, \quad \text{or} \quad y=\frac{-b^2}{a^2}\frac{x'}{y'}x.$$

Putting in this value of y in the equation to the ellipse, we get

$$b^2x^2+\frac{a^2b^4x'^2x^2}{a^4y'^2}=a^2b^2, \quad \text{or} \quad (b^2x'^2+a^2y'^2)\,x^2=a^4y'^2;$$

from which $x=\pm\frac{ay'}{b}$. Similarly, $y=\pm\frac{bx'}{a}$. The positive and negative signs show that the coordinates of the opposite extremities of the diameter are equal, but of opposite signs, as is evidently the case.

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* From the property here proved, that conjugate diameters bisect the chords parallel to each other, it follows obviously that the tangent at the extremity of a diameter is parallel to the conjugate diameter. For a chord drawn at that point parallel to the conjugate diameter is also bisected at that point, and must be therefore a tangent.

104. *The sum of the squares of a pair of conjugate semi-diameters is constant.*

Let the conjugate semi-diameters be termed respectively a' and b'. a' being the straight line from the origin to the point (x', y'), it follows that $a'^2 = x'^2 + y'^2$. Similarly, from above,

$$b'^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}.$$

Therefore

$$a'^2 + b'^2 = x'^2 + y'^2 + \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2} = \frac{a^2y'^2 + b^2x'^2}{b^2} + \frac{a^2y'^2 + b^2x'^2}{a^2}$$
$$= \frac{a^2b^2}{b^2} + \frac{a^2b^2}{a^2} = a^2 + b^2.$$

N.B.—The major and minor axes are a pair of conjugate diameters.

105. *The product of the focal radii of a point on the ellipse is equal to the square of the conjugate semi-diameter of the point (i.e., $rr' = b'^2$).*

For
$$b'^2 = a^2 + b^2 - a'^2 = a^2 + b^2 - x'^2 - y'^2$$
$$= a^2 + b^2 - x'^2 - \frac{b^2}{a^2}(a^2 - x'^2) = a^2 - x'^2 + \frac{b^2}{a^2}x'^2$$
$$= a^2 - \left(\frac{a^2 - b^2}{a^2}\right)x'^2 = a^2 - \frac{c^2}{a^2}x'^2 = a^2 - e^2x'^2 = rr'.$$

(Art. 82.)

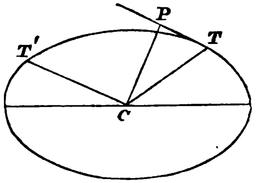
106. *The perpendicular from the centre on the tangent is equal to* $\frac{ab}{b'}$.

The perpendicular from the origin on the straight line $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$ is (Art. 24)

$$\frac{1}{\sqrt{\frac{x'^2}{a^4} + \frac{y'^2}{b^4}}} = \frac{ab}{\sqrt{\frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}}}.$$

But $b'^2 = \frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}$ (Art. 104). Therefore the perpendicular $p = \frac{ab}{b'}$.

107. *The angle between a pair of conjugate diameters is* $\sin^{-1} \frac{ab}{a'b'}$.



Let CT and CT' be a pair of conjugate semi-diameters, TP the tangent at T, parallel to CT', and CP the perpendicular (v) on it.

Now

$$\sin TCT' = \sin CTP = \frac{p}{a'} = \frac{ab}{a'b'}.$$

108. *The area of the triangle formed by a pair of conjugate semi-diameters, and the straight line joining their extremities, is constant.*

The triangle $TCT' = \frac{1}{2}a'.b' \sin TCT = \frac{1}{2}\frac{a'b'ab}{a'b'} = \frac{1}{2}ab$.

From this it evidently follows that the area of the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters, being equal to eight times the triangle, is equal to $4ab$.

It is evident that the diameters parallel to the straight lines joining the extremities of the major and minor axes are conjugate (Art. 102), and are equal; and that their inclinations to the axis of x are respectively

$$\tan^{-1} \frac{b}{a}, \text{ and } \tan^{-1} \frac{-b}{a},$$

and that the length of these semi-diameters is $\frac{1}{2}\sqrt{a^2+b^2}$.

109. Starting with the property of the ellipse that it is the locus of points the ratio of whose distance from a fixed point and a fixed straight line is constant and less than unity, its equation might have been found by a method

similar to that used in the case of the parabola, thus: Let the directrix be taken as axis of y, and the major axis as axis of x, and let the distance of the focus from the directrix be called p. Then, since $r=cTL$, and TL is equal to x, the abscissa of the point T; and calling its ordinate y, we have

since $r^2 = e^2TL^2$, $\quad y^2+(x-p^2) = e^2x^2$,

or $$y^2+(1-e^2)\,x^2-2px+p^2 = 0.$$

Let $e = 1$, that is, let $TF = TL$, which is the condition that the curve should be a parabola, and the coefficient of x in this equation vanishes, and it becomes $y^2 = 2px-p^2$, which (Art. 69) is the equation to a parabola. The parabola may therefore be considered a variety of the ellipse, its excentricity being unity.

The student can easily supply a figure.

EXERCISES ON THE EQUATION OF THE ELLIPSE.

[N.B.—Many of the properties and exercises for geometrical methods may be proved analytically, and form a useful exercise. — *E. g.*, Art. 87 and the geometrical example 11 follow instantly from the equation of the tangent.]

1. Find the excentricity of the ellipse $\frac{x^2}{7}+\frac{y^2}{5}=1$; and of the ellipse $\frac{2}{5}x^2+\frac{5}{8}y^2 = 50$. [*Ans.* $\sqrt{\frac{2}{7}}$; and $\frac{3}{5}$.

2. Transform the first of the above to the equal conjugate diameters as axes.

   [*Ans.* $x^2+y^2 = 6$. Why does not this represent a circle?

3. Shew that the equation to the tangent at the extremity of the Latus Rectum of an ellipse is $cx+ay = a^2$.

4. Find the equation to the normal at the extremity of the Latus Rectum of the ellipse $3x^2+4y^2 = 9$.

   [*Ans.* $y = 2x-\frac{\sqrt{3}}{4}$.

5. If r_1, r_2 are two semi-diameters at right angles to each other, show that $\frac{1}{r_1^2}+\frac{1}{r_2^2}$ is constant.

6. The chords joining the extremities of diameters at right angles touch a fixed circle.

7. An ellipse slides between two straight lines at right angles, find the locus of its centre.

8. Find the coordinates of a point on the ellipse such that the tangent at it makes equal angles with the axis.

[*Ans.* $x' = \frac{b^2}{\sqrt{a^2+b^2}}$, $y' = \frac{b^2}{\sqrt{a^2+b^2}}$.

9. $2y = 3x$ is a diameter of the ellipse $x^2+2y^2 = 4$, find the equation of the diameter conjugate to it.

[*Ans.* $9y+2x = 0$.

10. Of the angles between a tangent and diameter at the point of contact, the least is when the diameter is one of the equal conjugate diameters.

11. If a variable tangent meet two fixed parallel tangents, the product of the intercepts made on them will be equal to the square of the semi-diameter parallel to them.

12. If any tangent meet a pair of conjugate diameters, the product of the segments of it is equal to the square of the semi-diameter parallel to it.

13. Two equal rulers are jointed together, the extremity of one is fixed at a point, and the extremity of the other moves on a straight line through the point; find the equation to the curve described by any point on the second ruler.

[*Ans.* Let m be length of ruler, and n the distance of the point from the joint, the equation is

$$\frac{x^2}{(m+n)^2}+\frac{y^2}{(m-n)^2} = 1.$$

14. A circle is inscribed in the triangle formed by the

focal radii of any point on the ellipse and the major axis, find the locus of its centre.

> [*Ans.* Let (x', y') be the point on the ellipse, and (x, y) the centre of circle, then $y =$ radius of circle $= \dfrac{\text{area of triangle}}{\text{semi-perimeter}} = \dfrac{cy'}{a+c}$; also $x = s - r - c = ex$.
> Put in values of x' and y' in the equation
> $$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$
> and the result is an ellipse.

15. If two points be taken on the minor axis at a distance c from the centre, the sum of the squares of the perpendiculars from them on any tangent is constant.

CHAPTER IX.

THE HYPERBOLA.

110. THE simplest property of the hyperbola is that it is the locus of a point the difference of whose distances from two fixed points is constant. The two fixed points are called the *foci*. This is analogous to the definition from which the properties and equation of the ellipse were deduced.

The hyperbola may be defined as the locus of the vertex of a triangle of which the base and the difference of the sides are given; the extremities of the given base are the foci. From this we can easily see what the form of the curve must be. It will have two sides or branches similar to each other, as the greater side of the triangle, or longer radius vector of the curve, is measured from one focus or the other. As the difference, being constant, imposes no limit on the length of the sides of the triangle, each branch of the curve will go to infinity above and below the given base. Also, as the difference of the two sides of a triangle must be less than the third, the curve must pass between the foci.

The distance between the foci is called $2c$, and the given difference of the distances of any point on the curve from the foci is called $2a$. Thus the locus may be expressed $r-r'=2a$. The straight line joining the foci is called the *transverse axis* of the hyperbola, and the points where the curve cuts it are called the *vertices*, and the distance between is manifestly equal to $2a$. Also $\sqrt{c^2-a^2}=b$.

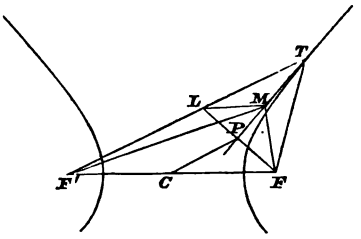
The ratio $\frac{c}{a}$ is called the *excentricity*, e, of the hyperbola, and is manifestly greater than unity.

The property of the focus and directrix will be shewn afterwards to exist in the hyperbola as in the ellipse; *i. e.*, that the ratio of the distances of a point on it from the focus and directrix is constant, $= e$. The three conic sections may therefore be summed up in one definition, the loci of points the ratio of whose distances from a fixed point (the focus) and from a fixed straight line (the directrix) is constant. Let that ratio $= e$; then, if e is less than unity, it is an ellipse; if $e = 1$, it is a parabola; if e is greater than unity, it is a hyperbola.

The properties of the ellipse closely resemble those of the hyperbola, and the proofs given in preceding Chapter can be repeated in this with very little change.

111. *The tangent at any point on the hyperbola is the straight line equally inclined to the focal radii of that point.*

Let T be any point on the hyperbola, of which F and F' are the foci. Draw TP equally inclined to the focal radii (*i. e.*, bisecting the angle between them); and it can be proved to be a tangent to the curve.



For it has one point T on the hyperbola, and any other point on it, as M, may be proved outside the hyperbola.

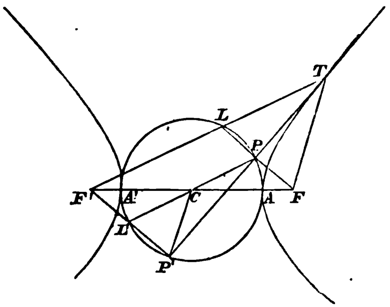
Draw FM and $F'M$, and take L on $F'T$ so that $TL = TF$ and $F'L = F'T - FT = 2a$; join FL and ML. Since $TL = TF$, and T bisecting FTL is common to both triangles FTP and LTP, they are equal in every respect; and $MF' - ML = MF' - MF$. But $MF' - ML < F'L$; therefore $MF' - MF$ is less than $2a$; therefore M is outside the curve. Consequently TP, having one point on the curve and every other point outside the curve, is a tangent.

112. *To find the locus of the foot of the perpendicular from the focus on the tangent.*

Since P is middle point of FL, and C middle point of FF', it follows that $CP = \frac{1}{2}F'L = a$; therefore the locus of P is a circle with centre at C and radius equal to a.

113. *The product of the perpendiculars from the foci on the tangent is equal to* b^2.

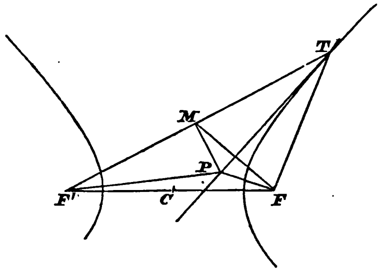
This is proved, as in the case of the ellipse, by means of the circle in foregoing proposition. The perpendiculars FP and $F'P'$ meet the tangent on its circumference. Produce PC to meet $P'F'$ in L; this point must also be on the circle, as the angle $PP'F'$ is right. The triangles $F'CL'$ and FCP are manifestly equal (Euc., Book I., Prop. 4); therefore $F'L = FP$. Therefore



$$FP \times F'P' = F'P' \times F'L' = F'A' \times F'A$$
$$= (c+a)(c-a) = c^2-a^2 = b^2.$$

114. *If a perpendicular be erected at the focus to the radius vector of any point, and produced to meet the tangent at that point, the locus of their intersection will be a straight line.*

Let T be any point on the hyperbola, and TP the tangent at it, P being its intersection with FP, the perpendicular at F to FT, the radius vector of T. On the other radius vector take M so that $TM = TF$; join PM. The triangles PTF and PTM are manifestly equal in every respect (Euc., Book I., Prop. 4); consequently $PM = PF$, and the angle $PMT = PFT$, and is a right angle. Therefore

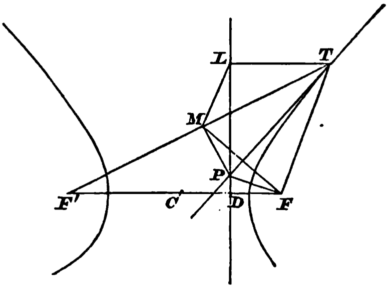


$$F'P^2 - PM^2 \;(= F'P^2 - FP^2) = FM^2 = (r - r')^2 = 4a^2.$$

Since the difference of the squares of the distances of the point P from F and F' is constant, the locus of it is a straight line perpendicular to FF' (Euc., Book II., Prop. 13, Cor.).

This straight line is called the *directrix ;* and there are evidently two directrices, corresponding to the two foci.

115. *The ratio of the distances of any point on the hyperbola from the focus and directrix is constant* $(= e)$.



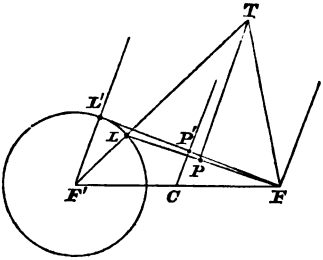
Let the same construction remain; and draw TL perpendicular to the directrix DP, and join MF and ML. Because the angles TFP, TMP, and TLP are right, a circle on TP as diameter will pass through F, M, and L; consequently the angle PFM is equal to the angle PLM. But $PFM = PMF$ (as $PM = PF$; therefore the angle PLM is equal to the angle PMF. Also, since the angles TLP and $F'MP$ are right. the angle FMF' is equal to the angle TLM; and since TL is parallel to FF', the angle $F'TL$ is equal to $FF'M$. Therefore the triangles $FF'M$ and TML are similar. Therefore

$$TM : TL = FF' : FM, \quad \text{or} \quad r : TL = 2c : 2a.$$

Therefore $$r = \frac{c}{a}\, TL = e\,TL.$$

116. From the tangent being equally inclined to the focal distances, and their differences being constant, a series of points on the hyperbola, and tangents at them, can be easily found by the following construction:—With one

focus as centre, and a radius $= 2a$, describe a circle; join any point L on the circumference of this circle with the other focus, and at the middle point (P) of this joining line erect a perpendicular PT, and the point T, where the radius $F'L$ produced meets it, will be on the hyperbola, and the straight line PT a tangent at that point. For the triangles TPF and TPL are equal; therefore the difference of the distances of T from F and F' $= F'L = 2a$; and PT is equally inclined to the focal radii, and is a tangent.



Let us take the case where the angle $FL'F'$ is right; then the perpendicular from the middle point of FL' will be parallel to the radius $F'L'$ produced; and it will also pass through the centre C of the hyperbola, being the middle point of FF'. It is manifestly a tangent to the curve, meeting it at an infinite distance; the point T has, as it were, moved off to infinity.

From this construction it is manifest that

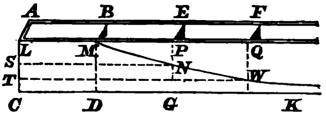
$$FF'^2 = FL'^2 + L'F'^2, \quad i.e., \quad 4c^2 = 4a^2 + LF'^2;$$

therefore $\quad L'F' = 2b, \quad$ or $\quad F'P = b.$

From this evidently $\quad \tan FCP = \dfrac{FP}{CP} = \dfrac{b}{a}.$

It is equally clear that the straight line through the centre at an inclination $\tan^{-1} \dfrac{-b}{a}$ will be a tangent at infinity to the other branch of the hyperbola; and each produced below the transverse axis will be a tangent at infinity to the opposite branch. These tangents to the hyperbola at infinity are called "asymptotes" (from ἀσυμπίπτω, not to fall together), because, while continually approaching the curve, they never meet it except at infinity.

117. The idea of an asymptote is sometimes not easily conceived by a beginner. The following illustration may help to make it more clear:—Suppose a vessel in the form of a rectangular parallelepiped ($ALCK$), whose sides and bottom were prolonged indefinitely in one direction AF, CK, and that one side BMD could be moved along always parallel to itself. If it was filled with water, and then this side moved to the position EPG, making $LM = MP$, the water would have sunk down to one-half the original level, its corner making on the side some such trace as indicated by the curved line. If the moveable end was brought still farther on, the level of the water must become lower, and the curved trace would be continued; but supposing the water to be a perfect fluid, and having no adhesion to the sides of the vessel, it is evident that there must always be a layer of water on the bottom of the vessel or trough, and the curved trace would be for ever approaching though never reaching the straight line CK. This line would be an asymptote to the curved trace. It will be afterwards seen that the curve so traced would be a rectangular hyperbola.



118. *If from any point on a hyperbola a straight line be drawn parallel to the asymptote to meet the directrix, it will be equal to the focal distance of the point.*

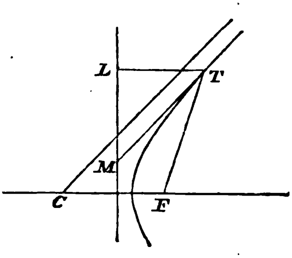
For $$TM = TL \times \sec MTL = TL \sec \theta,$$

where $\theta = \tan^{-1}\dfrac{b}{a}$, the angle which the asymptote makes with the axis of x; therefore

$$TM = TL\sqrt{1+\tan^2\theta}$$

$$= TL\frac{\sqrt{a^2+b^2}}{a} = \frac{c}{a}TL$$

$$= eTL = r.$$



From this an instrument for drawing the hyperbola by continuous motion has been devised. A ruler, bent at an angle equal to $\left(\pi - \tan^{-1}\frac{b}{a}\right)$, slides with one side of the angle along a straight line, which is the directrix; and a string equal in length to the other side is fastened to its extremity, and stretched along it by a ring or traveller carrying a pencil, the other end of the string being fastened at the focus. The length of the string being always equal to the length of the leg of the ruler, it is manifest that the point carrying the pencil will be always equally distant from the focus and from the directrix measured along that leg of the ruler which is parallel to the asymptote. If the angle at which the ruler is bent is right, the curve is a parabola.

EQUATION OF THE HYPERBOLA.

119. The equation to the hyperbola can be found in the same manner as the equation to the ellipse, taking the corresponding property of the curve—viz., that the difference of the focal distances of any point on it is constant; that is, $r - r' = 2a$.

Take the straight line joining the foci as axis of x, and a straight line perpendicular to it at the centre as axis of y; let x, y be the coordinates of any point on the curve, and the focal distances of the point r and r'; then

$$r = \sqrt{y^2 + (x+c)^2}, \quad \text{and} \quad r' = \sqrt{y^2 + (x-c)^2};$$

therefore $r - r' = 2a = \sqrt{y^2 + (x+c)^2} - \sqrt{y^2 + (x-c)^2}$.

Proceeding as in Article 91, and for c^2 substituting $a^2 + b^2$, we get

$$b^2x^2 - a^2y^2 = a^2b^2, \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This is the simplest form of the equation to the hyperbola.

Although the curve does not meet the axis of y, yet if the length b $(= \sqrt{c^2-a^2})$ be taken upon it above and below the centre, it will be hereafter seen that this, which is called the *conjugate axis*, has important relations with the hyperbola.

By exactly the same process as in the ellipse (Art. 93), the equations of the tangent and normal to the hyperbola are found to be respectively

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1, \quad \text{and} \quad y-y'=-\frac{a^2y'}{b^2x'}(x-x').$$

120. *The focal distances of any point on the hyperbola are respectively* $ex+a$ *and* $ex-a$.

For $r^2=y^2+(x+c)^2$, and $r'^2=y^2+(x-c)^2$.

Therefore $r^2-r'^2=4cx.$

Therefore $(r+r')(r-r')=4cx.$

Therefore $(r+r')2a=4cx$, or $r+r'=2\frac{c}{a}x=2ex.$

Since the sum of the radii vectores $=2ex$, and their difference $=2a$, they must be $ex+a$ and $ex-a$.

If the property proved geometrically in Article 115, be taken as definition, that the hyperbola is the locus of points whose distances from a fixed point and from a fixed straight line are in a constant ratio, that ratio being greater than unity,—*i. e.*, the point being nearer the straight line, the directrix, than the fixed point, the focus,—the equation can be easily proved. Taking the directrix as axis of y, and the straight line perpendicular to it through the foci as axis of x, and letting the given ratio $e=\frac{\sqrt{a^2+b^2}}{a}$, and then transferring the origin to the centre, it will become, as before,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

121. The polar equation to the hyperbola, when the centre is origin, and the transverse axis (*i. e.*, the straight

line joining the foci) is initial line, is found by substituting for x and y; thus,

$$r^2(b^2\cos^2\theta - a^2\sin^2\theta) = a^2b^2, \quad \text{or} \quad \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{1}{r^2}.$$

When the focus is origin, we have

$r = eTL = e(DF + FN)$ (see Fig., Art. 115, TN being the ordinate)

$$= e(CF - CD + FN) = e\left(ea - \frac{a}{e} + r\cos\theta\right);$$

therefore $r = a(e^2 - 1) + er\cos\theta$, or $r = \dfrac{a(e^2-1)}{1 - e\cos\theta}$.

122. *If p be the length of the perpendicular from the centre on the tangent, and a its inclination to the principal axis,*

$$p^2 = a^2\cos^2 a - b^2\sin^2 a.$$

From Articles 12 and 119,

$$x\cos a + y\sin a = p \quad \text{and} \quad \frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$$

must both represent the tangent. Therefore, the coefficients being proportional,

$$\frac{\cos a}{p} = \frac{x'}{a^2} \quad \text{and} \quad \frac{\sin a}{p} = \frac{-y'}{b^2}.$$

Consequently

$$\frac{a\cos a}{p} = \frac{x'}{a} \quad \text{and} \quad \frac{b\sin a}{p} = \frac{-y'}{b}.$$

Squaring and subtracting, we have

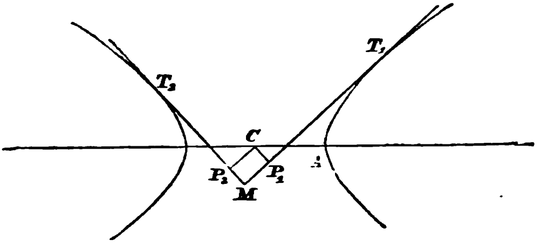
$$\frac{a^2\cos^2 a}{p^2} - \frac{b^2\sin^2 a}{p^2} = \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1,$$

since (x', y'), the point of contact, is on the hyperbola.

Therefore $\quad a^2\cos^2 a - b^2\sin^2 a = p^2.$

123. *To find the locus of intersection of two tangents that are at right angles to each other.*

Let MT_1 and MT_2 be two tangents at right angles, and let the angle ACP_1 be α, and the angle ACP_2 be $\alpha+90°$.



Then $\qquad (CP_1)^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$;

also $\qquad (CP_2)^2 = a^2\sin^2\alpha - b^2\cos^2\alpha$,

adding, $\qquad (CP_1)^2+(CP_2)^2 = a^2-b^2 = (P_1P_2)^2$.

But as CP_1MP_2 is a rectangle, $CM = P_1P_2$. Consequently M is at a distance from $C = \sqrt{a^2-b^2}$, and its locus is a circle with its centre at C and its radius $= \sqrt{a^2-b^2}$.

From this it will be seen that, if $a=b$, the circle becomes a point, and in that case but one pair of tangents at right angles can be drawn to the hyperbola.

If $a<b$, then the locus is imaginary, and the hyperbola can have no pair of tangents at right angles.

CONJUGATE DIAMETERS; CONJUGATE HYPERBOLA; ASYMPTOTES.

124. Most of the properties of the conjugate diameters of the hyperbola can be deduced as in the case of the ellipse. Articles 99 to 101 are directly applicable to

the hyperbola; remembering that b^2 has a negative sign, and taking a different figure.

Consequently, if (x', y') be a point on a hyperbola, $x'y = y'x$ is the diameter through it, and

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0$$

is the conjugate diameter, and conjugate diameters bisect the chords parallel to each other. Also, if θ be the angle made by a diameter with the axis of x, and θ' the angle made by the conjugate diameter with the axis of x,

$$\tan\theta \times \tan\theta' = \frac{b^2}{a^2}.$$

It will be a useful exercise to work out the equations of the chords parallel to the diameters so as to find the coordinates of the middle points, as done in the case of the ellipse, and to notice what changes must be made in consequence of the negative sign of the coefficient of y.

125. *Every diameter (i.e., straight line passing through the centre) does not necessarily meet the hyperbola.*

This is obvious from the figure, as it is not a closed curve; but may be proved analytically.

Let $y = mx$: putting in this value of y in the equation to the hyperbola, we get

$$b^2x^2 - a^2m^2x^2 = a^2b^2;$$

from this $x = \frac{ab}{\sqrt{(b^2 - a^2m^2)}}$. Consequently, if m^2 be less than $\frac{b^2}{a^2}$, we have possible values of x; but if it be greater, we have impossible values. Therefore, when m is greater than $\frac{b}{a}$, or when the inclination of the straight line to the axis of x is greater than $\tan^{-1}\frac{b}{a}$, the straight line does not meet the curve.

When $m = \frac{b}{a}$ the value of x becomes infinite; so that the two straight lines that can be drawn at the inclination $\tan^{-1}\frac{b}{a}$ to the axis of x meet the curve only at an infinite distance.

126. We will now proceed to investigate the conjugate hyperbola. If the signs of the equation to the original hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be changed on one side only, so as to become $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get the equation to a hyperbola which manifestly bears the same relation to the axis of y that the original hyperbola has to the axis of x. This new hyperbola is called the conjugate hyperbola to the former; its foci are situated on the axis of y at a distance the same as in the original hyperbola, for in each case $c = \sqrt{a^2+b^2}$; but the distance between its vertices is $2b$, and the difference of the focal radii of any point on it is manifestly $2b$.

127. *The diameter conjugate to that drawn from any point on the hyperbola meets the conjugate hyperbola.*

Let (x', y') be a point on the original hyperbola; the equation of the diameter conjugate to that passing through the point is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0$. Combining this with the equation to the conjugate hyperbola $a^2y^2 - b^2x^2 = a^2b^2$, we find the value of x to be $\pm\frac{ay'}{b}$, and of y $\pm\frac{bx'}{a}$.

If the length of a semi-diameter from any point on the original hyperbola be called a', the length of the conjugate semi-diameter meeting the conjugate hyperbola is called b'.

128. *The difference of the squares of two conjugate semi-diameters is constant.*

For $a'^2 = x'^2 + y'^2$ and $b'^2 = \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}$,

therefore

$$a'^2 - b'^2 = x'^2 + y'^2 - \frac{a^2y'^2}{b^2} - \frac{b^2x'^2}{a^2}$$

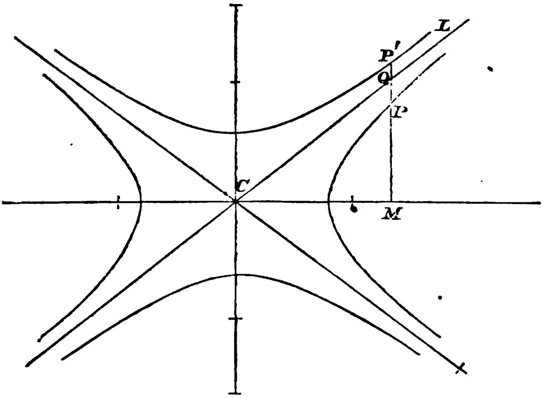
$$= \frac{b^2x'^2 - a^2y'^2}{b^2} + \frac{a^2y'^2 - b^2x'^2}{a^2} = \frac{a^2b^2}{b^2} + \frac{-a^2b^2}{b^2} = a^2 - b^2.$$

129. In precisely the same manner as used in the ellipse (Arts. 106 and 107) it can be proved that the perpendicular from the centre on the tangent $p = \frac{ab}{b'}$; and from this that the sine of the angle between a pair of conjugate diameters is $\frac{ab}{a'b}$. Also that the triangle formed by a pair of conjugate semi-diameters and the straight line joining their extremities is of constant area, and equal to $\frac{1}{2}ab$; and the area of the parallelogram formed by the tangents at the extremities of a pair of conjugate diameters (*i.e.*, the tangents to the hyperbola and the conjugate hyperbola) is equal to $4ab$.

ASYMPTOTES.

130. We have now to consider the case of the straight lines drawn through the centre at an inclination to the axis of x whose tangent is $\pm \frac{b}{a}$. In Art. 116, it was shown that these straight lines would never meet the curve except when the value of x was infinite, that is, at an infinite distance from the centre.

It can further be shewn that the curve is continually approaching these straight lines, although never meeting them. Let Q be a point on the straight line $y = \frac{b}{a}x$, and



P the point where the ordinate of Q meets the hyperbola

$$\frac{x^2}{\quad} - \frac{y^2}{b^2} = 1.$$

Let QPM be that ordinate; then

$$QM = \frac{bx}{a} \text{ and } PM = \frac{b}{a}(x^2-a^2)^{\frac{1}{2}}.$$

Therefore $QP = QM - PM = \frac{b}{a}\,\frac{x-(x^2-a^2)^{\frac{1}{2}}}{a}.$

Multiplying numerator and denominator of this fraction by $x+(x^2-a^2)^{\frac{1}{2}}$, it becomes $\frac{b}{a}\,\frac{a^2}{x+(x^2-a^2)^{\frac{1}{2}}}$. Therefore $QP = \frac{ab}{x+(x^2-a^2)^{\frac{1}{2}}}$. From this it is evident that, as the point Q is taken farther along the straight line CQ, and its abscissa becomes greater, that the value of QP diminishes; in other words, that P continually approaches Q; but QP

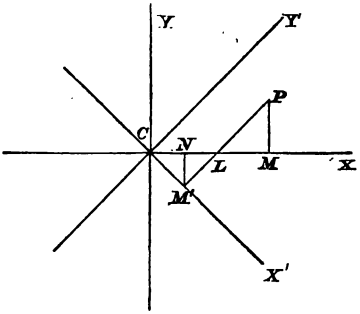
does not vanish unless x is infinite, then only does the curve meet its asymptote.

The equation to one asymptote is $y = \frac{b}{a}x$, and to the other $y = -\frac{b}{a}x$; both asymptotes are included in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$. It is evident that the hyperbola and its conjugate have the same asymptotes. For, let QM (see figure) be produced to meet the conjugate hyperbola in P', then

$$P'Q = P'M - QM = \frac{b}{a}\left[(x^2+a^2)^{\frac{1}{2}} - x\right] = \frac{ab}{(x^2+a^2)^{\frac{1}{2}}+x},$$

which diminishes indefinitely as x is increased.

131. The equation to the hyperbola takes a very important form when transformed to the asymptotes as axes.



Let P be any point on the curve whose coordinates are PM and CM referred to the original axes, and PM' and CM' referred to the new. Draw $M'N$ perpendicular to CM; then we have

$$\begin{aligned} CM &= CN + NM \\ &= CM' \cos M'CN \\ &\quad + PM' \cos M'CN \\ &= (CM' + PM') \cos M'CN. \end{aligned}$$

Again, $PM = PL \sin PLM = (PM' - M'L) \sin CLM'$
$= (PM' - CM') \sin M'CN,$

since PM' is parallel to CY'.

But $\tan MCN = \frac{b}{a}$, therefore

$$\sin M'CN = \frac{b}{\sqrt{a^2+b^2}}, \text{ and } \cos MCN = \frac{a}{\sqrt{a^2+b^2}};$$

also CM and MP are x and y referred to original axes, and CM' and PM' are x and y referred to the new axes. Consequently for x we must substitute $(x+y)\frac{a}{\sqrt{a^2+b^2}}$, and for y we must substitute $(y-x)\frac{b}{\sqrt{a^2+b^2}}$, to transform the equation to the new axes. Making these substitutions in

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we have $$\frac{(x+y)^2}{a^2+b^2} - \frac{(x-y)^2}{a^2+b^2} = 1,$$

or $$4xy = a^2+b^2 \text{ or } xy = \tfrac{1}{4}a^2+b^2.$$

The hyperbola may therefore be defined as the locus of a point such that the product of the straight lines drawn from it parallel to two given intersecting straight lines is constant. It is manifest that these intersecting straight lines are asymptotes, because since the product is constant, if the straight line from the point to one of them is nothing, the straight line to the other must be infinite.

132. $xy = -\frac{1}{4}(a^2+b^2)$ is manifestly the equation to the conjugate hyperbola referred to the asymptotes.

133. *To find the equation to the tangent to the hyperbola when referred to the asymptotes as axes.*

The equation to the straight line joining any two points on the curve is $y - y' = \frac{y'' - y'}{x'' - x'}(x - x')$;

putting the condition that both are on the curve, viz., that $x'y' = x''y''$, since each is equal to $\frac{1}{4}(a^2+b^2)$, we have

$$y'' = \frac{x'y'}{x''};$$

therefore the equation to the secant may be written

$$y-y' = \frac{\frac{x'y'}{x''} - y'}{x''-x'}(x-x');$$

or $$y-y' = \frac{y'(x'-x'')}{x''(x''-x')}(x-x'),$$

or, changing sign of denominator and cancelling,

$$y-y' = -\frac{y'}{x''}(x-x');$$

when the points coincide and the secant becomes a tangent,

$$y-y' = \frac{y'}{x'}(x-x'),$$

which gives $x'y+y'x = 2x'y' = \frac{1}{2}(a^2+b^2)$.

From this it is manifest that the intercept on the tangent between the asymptotes is bisected at the point of contact. For in the equation $x'y+y'x = 2x'y'$, if $y=0$, $x=2x'$; and if $x=0$, $y=2y'$.

From this it follows that the area of the triangle made by the tangent and asymptotes, being twice the parallelogram formed by the coordinates of the point of contact, is constant and equal to $\frac{1}{2}(a^2+b^2)$.

Hence the hyperbola might be defined as the envelope of a straight line which cuts off triangles of constant area with two intersecting straight lines. Or it may be thus expressed: Given vertical angle and area of a triangle, the base must touch a hyperbola of which the sides of the vertical angle are asymptotes. Or again, since the point of contact is the middle point of the intercept or tangent: Given vertical angle and area of a triangle, the locus of the middle point of base is a hyperbola.

EQUILATERAL OR RECTANGULAR HYPERBOLA.

134. If, in the equation to the hyperbola, a is made equal to b, we have $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$, or $x^2-y^2=a^2$. This is called the equilateral hyperbola from the equality of the axes; and it is manifestly identical except in position with its conjugate hyperbola.

Further, the asymptotes are manifestly at right angles, from which it is called the rectangular hyperbola (see Art. 117); and its equation referred to the asymptotes is $xy=\frac{1}{2}a^2$.

In the ellipse, if $a=b$, the equation becomes a circle; thus the equilateral hyperbola bears the same relation to the general equation to the hyperbola that a circle does to the ellipse.

Referring to Art. 123, it will be seen that, when the locus of points from which tangents at right angles can be drawn becomes imaginary, *i.e.* when $a<b$, then the hyperbola can have no tangents at right angles, but the conjugate hyperbola can; and, *vice versâ*, when the locus becomes a point, *i.e.* when $a=b$, then the hyperbola can have one and only one pair of tangents at right angles, *i.e.* the asymptotes, and is a rectangular hyperbola.

GEOMETRICAL EXERCISES ON THE HYPERBOLA.

1. If an ellipse and hyperbola have the same foci, the tangents at their intersections are at right angles.

2. Shew that the locus of the centres of circles which touch two given circles is an ellipse and a hyperbola.

3. The chord of contact of any point on the directrix passes through the focus.

4. A circle on the distance between the foci as diameter passes through the intersection of the tangents at the vertices and the asymptotes; and the auxiliary circle passes through the intersection of the diameters and asymptotes.

5. If from any point a pair of tangents be drawn to the same branch of a hyperbola, the angles subtended by them at the focus are equal, if to the different branches the angles subtended at the foci are supplemental.

6. If a circle be inscribed in the triangle formed by the focal radii of any point and the straight line joining the foci, the locus of its centre is the tangent at the vertex.

7. Given an arc of a conic section, shew how to find whether it is a parabola, an ellipse, or a hyperbola.

8. A rifle bullet hits a target; find the locus of places at which the sound of the discharge and of the shot are heard together.

EXERCISES ON THE EQUATION TO THE HYPERBOLA.

N.B.—Most of the geometrical exercises can also be done analytically; also the properties and exercises given for the ellipse are generally applicable to the hyperbola.

1. Given the transverse axis of a hyperbola, and that it bisects the distances between the centre and the foci, find its equation.

[*Ans.* $3x^2 - y^2 = 3a^2$ ($2a$ being the transverse axis).

2. If the ordinate of any point on a hyperbola be produced till it is equal to the focal distance of the point, find the locus of its extremity.

3. The extremities of any chord of an ellipse parallel to the minor axis are joined with the extremities of the major axis; shew that the locus of the intersections of the joining lines will be a hyperbola.

4. If the extremities of parallel chords in a circle be joined with the extremities of any diameter, prove that the joining axis will intersect on a rectangular hyperbola.

5. The radius of the circle that touches a hyperbola and its asymptotes is equal to the intercept on the latus rectum produced between the curve and its asymptote.

CHAPTER XI.

RELATIONS OF THE CONIC SECTIONS TO THE CONE.

135. The curves treated of in the foregoing pages are called the Conic Sections, because they can be formed by a plane suitably cutting a cone. This may be easily exemplified by a sheet of paper and a cone of light or of shadow. It is a remarkable coincidence that a point, a straight line, two straight lines, a circle, a parabola, an ellipse, a hyperbola, some one of which must be represented by the general equation of the second degree, can also be obtained from a plane cutting a cone.

We will now proceed to the proof of this, deducing the fundamental properties of the conic sections from the figure of the cone; and showing what relation the focus and directrix of a parabola, ellipse, and hyperbola have to the cone from which they are cut.

A cone may be conceived as generated by the revolution of a straight line round a fixed intersecting straight line. The fixed straight line is called the *axis* of the cone, and the inclination of the straight lines is called the *generating angle*, and the point where they intersect the *vertex* of the cone.

The section made by a plane at right angles to the axis is manifestly a circle, the axis passing through the centre; and every point on the circumference is equally distant from the vertex of the cone. Also the intercepts between two circular sections of a cone on the straight lines drawn on its surface through the vertex, or on different positions of the generating straight line, are equal. Also it is easily conceived that a sphere can be inscribed in a cone (as a bullet or marble dropped into the extinguisher of a candle), and that it will touch the cone in a circle.

136. The following propositions may now be enunciated:

The focus of a conic section is the point in which the inscribed sphere touches the cutting plane.

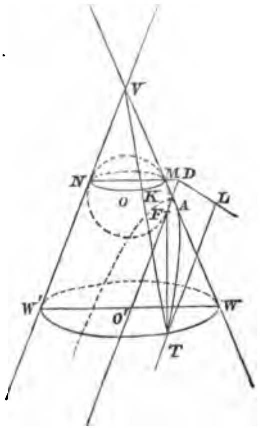
The directrix of a conic section is the straight line formed by the intersection of the plane of the circle in which the inscribed sphere touches the cone, with the cutting plane.

137. We will proceed to prove these first in the case of the parabola.

If the plane cutting a cone is parallel to a position of the generating line, or, in other words, if the angle at which the cutting plane is inclined to the axis of the cone is equal to the generating angle of the cone, then the section will extend to infinities, as the plane will manifestly not intersect the cone.

The section will be a parabola.

Let O be the centre of the sphere inscribed in the cone, touching the cutting plane in F. (In the figure, the cutting plane is supposed to be perpendicular to the plane of the paper.)

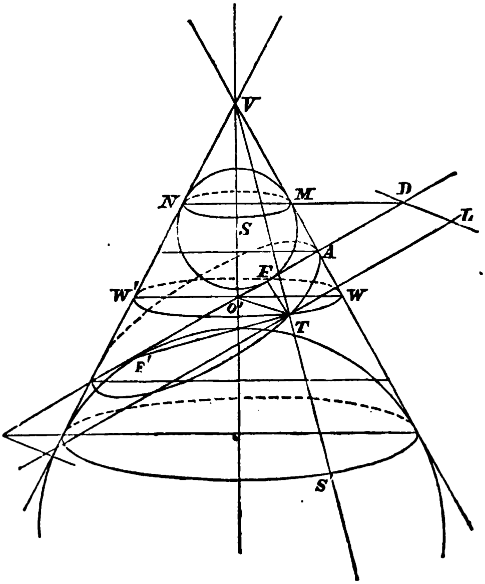


The straight line FD formed by the intersection of the cutting plane and plane of the paper will be perpendicular to DL, the straight line formed by the intersection of the cutting plane with the plane of the circle in which the inscribed sphere touches the cone. Take any point T on the parabola, and draw TL perpendicular to DL; and join TF and TV with the vertex of the cone meeting the circle MKN in K; and let WTW' be the circular section of the cone through T, whose plane is at right angles to the axis of the cone.

TF is equal to TK, because they are tangents to the sphere from the same point. Also, since TO'

and *LD* are parallel, being sections of two parallel planes by the cutting plane, and *TC* and *O′D* are parallel, being both perpendicular to *DL*, *TL* is equal to *O′D*; also *TK* is equal to *WM*, both being the distance between two circles on the cone. Consequently *TF* is to *TL* as *WM* is to *O′D*; that is, as *AM* to *AD*; that is, as the sine of the angle *AMD* to the sine of the angle *ADM*. But these angles are equal, one being the complement of the generating angle of the cone, and the other the complement of the angle at which the cutting plane is inclined to the axis of the cone. Therefore the distances of any point on the parabola from *F*, and the straight line *DL*, are equal.

138. In the ellipse, the cutting plane and cone can manifestly be touched by two spheres, one between the cutting plane and the vertex of the cone, and the other touching



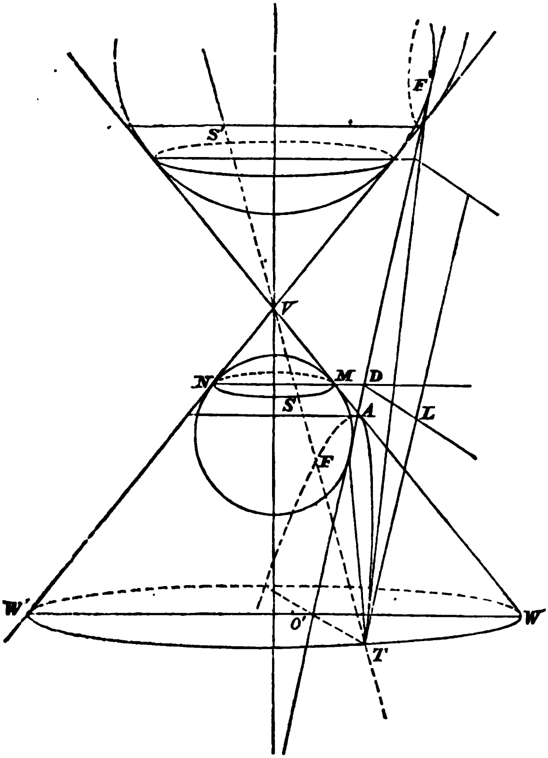
the cutting plane on the lower side. Let the points of contact be F and F'; then, taking any point T on the ellipse, and joining TF and TF', and drawing TS' through T and the vertex of the cone, we have $TF=TS$ and $TF'=TS'$ as tangents each to a sphere from the same point; consequently $TF+TF'=SS'$; but this is the distance between the two circles on the cone in which the inscribed spheres touch it, and is therefore constant. This proves the property $r+r'=2a$.

The property of the directrix can be proved as for the parabola. Let the plane of the circle MSN be produced to meet the cutting plane in DL, and let WTW' be a circular section of the cone through T. Then $TF=TS$, being tangents to the same sphere; and $TS=MW$, being the distance between two circles on the cone. Also $TL=O'D$ ($TLDO'$ being a rectangle). Consequently TF is to TL as WM is to $O'D$; that is, as AM to AD; that is, as the sine of the angle ADM to the sine of the angle AMD. But this is a constant ratio, one being the complement of the generating angle of the cone, the other the complement of the angle at which the cutting plane is inclined to the axis of the cone.

The excentricity of the ellipse is equal to $\dfrac{TF}{TL}=\dfrac{\cos\theta}{\cos\alpha}$, where α is the generating angle of the cone, and θ the angle at which the cutting plane is inclined to the axis of the cone. From this it follows that an ellipse of any excentricity can be cut from any cone; for, whatever the value of $\cos\alpha$, θ can be found so that $\cos\theta$ is any submultiple of it, and therefore $\dfrac{\cos\theta}{\cos\alpha}$ can represent any fraction.

139. The elementary properties of the hyperbola can be proved from the cone as in the ellipse. In this the cutting plane will meet the cone above the vertex, and so form the two branches of the curve; and the inscribed spheres will touch the cutting plane one above and the other below the vertex, as in the figure. Let F and F' be the points of contact; then, taking any point T on the hyperbola, and join-

ing TF and TF', and drawing TV through T and the vertex of the cone and producing it on the cone above the



vertex, we have $TF=TS$ and $TF'=TS'$ as pairs of tangents each to a sphere from the same point. Consequently $TF-TF'=SS'$; but this is the distance between the two circles on the cone in which the inscribed spheres touch, and is therefore the same, whatever be the position of T. This proves the property $r-r'=2a$.

If the plane of the circle MSN be produced to meet the cutting plane, the straight line DL formed by their intersection will be the directrix of the hyperbola. Let WTW' be a circular section of the cone through T; then $TF=TS$, being tangents to the same sphere, and $TS=WM$, being the distance between two circles on the cone. Also $TL=O'D$ (since $TLDO'$ is a rectangle); consequently TF is to TL as WM is to $O'D$; that is, as AM to AD; that is, as the sine of the angle ADM is to the sine of the angle AMD. This is a constant ratio, being that of the cosine of the angle at which the plane is inclined to the axis of the cone to the cosine of the generating angle of the cone.

From this it follows that a hyperbola of any excentricity cannot be cut from any cone; the excentricity being equal to $\frac{\cos\theta}{\cos\alpha}$ and greater than unity: as no cosine can be greater than one, when $\cos\alpha$ is given it is not possible to make $\cos\theta$ any multiple of it.

140. If the plane which cuts the cone so as to form a parabola be supposed to move through the cone parallel to itself, it will form a series of parabolas of less and less parameter* as it approaches the vertex; and when it passes through the vertex it will touch the cone along a generating line. The parabola will have shrunk into a straight line, or rather two coincident straight lines.

141. If the plane which cuts the cone so as to form an ellipse have its inclination altered so as to be at right angles to the axis, the ellipse will become a circle. If the plane be moved so as to pass through the vertex, the ellipse will become a point.

142. If the plane which cuts the cone so as to form a hyperbola be moved parallel to itself, it will form a series of hyperbolas, which, when the plane passes through the vertex, will become two intersecting straight lines. It will easily be seen that these represent the asymptotes to the series of hyperbolas.

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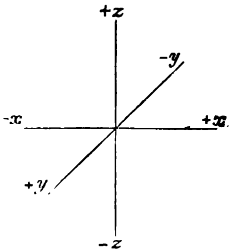
* In the equation $y^2=4ax$, a is called the parameter, and is equal to half the distance from the focus to the directrix. From Fig. to Art. 137 it is easily seen to be equal to $r\sin\alpha$, where r is the radius of inscribed sphere, and α is generating angle of the cone.

CHAPTER XII.

FIRST PRINCIPLES OF ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

143. It is easy to see that the method applied to denoting the position of a point in a plane may be extended to the position of a point in space. Let us suppose three straight lines in space meeting in a point which we will call the *origin* (the corner of a room is an illustration), and the three straight lines are termed the *axes* of x, y, and z respectively. The three planes which pass through each pair of these axes (like the floor and sides of a room) are called the *coordinate planes;* that which passes through the axes of x and y being called the plane of xy, that which passes through the axes of x and z being called the plane of xz, and the third being the plane of yz, passing through the axes of y and z. Now, if from any point in space a straight line be drawn parallel to the axis of z to meet the plane of xy, that line is called the z of the point; also the position of the point in which this line parallel to the axis of z meets the plane of xy can be denoted by the values of its distances from the axes of x and y in that plane (Art. 2). These three lines, or rather lengths, are called the x, the y, and the z of the point. The first two, the x and y, being given, enable us to find the point in the plane of xy in which the straight line parallel to the axis of z meets it; proceeding from this point in the direction of z the given length for z, we arrive at the point whose coordinates are given.

The three axes, or rather the three coordinate planes through them, divide space around the origin into eight regions. When the axes are rectangular they are called octants, and may be easily understood by supposing the origin to be at the centre of a sphere; the axes will then be three diameters at right angles, and the coordinate planes through them will divide the surface of the sphere into eight equal quadrantal spherical triangles (*i.e.*, triangles each of whose sides is a quadrant of a circle): supposing these axes and the coordinate planes prolonged indefinitely, they will divide space into eight octants.



The convention for the signs of coordinates is exactly the same as in plane coordinate geometry.

Starting with one octant, the coordinates of any point in which are taken as positive, when in any of the other octants a coordinate is measured along the axes in the same direction as in the first octant its sign is positive, but when in an opposite direction it is negative; and the eight permutations of positive and negative signs to the three coordinates correspond to the eight octants.

Or we may consider it in this way:—Let the axis of z be vertical; then, when z has a positive value, the position of a point above the plane of xy can be denoted in any of the four quadrants in the plane of xy by the positive and negative signs of x and y, as in Art. 3; and the positive value of z gives four corresponding points above that plane. Let the value of z be negative, and we have four corresponding points under the plane of xy. Thus, by giving the proper signs to x, y, and z, the position of a point can be denoted in any of the eight octants around the origin.

144. We can now proceed to the interpretation of an equation in x, y, and z. Let $f(xyz) = 0$ be an equation involving some function of x, y, and z. This may be written $z = \phi(xy)$, where $\varphi(xy)$ is some other function of x and y.

From this it is plain that any pair of values of x and y give a corresponding value of z. That is, for any point in the plane of xy we get a corresponding point above or below it in the direction of the axis of z. Also any change in one of the variables involves some corresponding change in the other two. Giving every possible value to x and y, we get a series of values of z; that is, for every point in the plane of xy we get some corresponding point in the direction of z. This assemblage of points constitutes a surface. Consequently $f(xyz) = 0$ represents a surface, as $f(xy)$ represents a line in the plane of xy. It is possible that the equation $z = \phi(xy)$ may give several values of z for any pair of values of x and y. In that case the equation $f(xyz)$ may represent a *surface of several sheets*, as $f(xy)$ may in plane coordinate geometry represent a locus consisting of several lines.

The equation $(x-a)^2+(y-b)^2+(z-c)^2 = 0$ manifestly represents a point; for it can only be satisfied by $x=a$, $y=b$, and $z=c$.

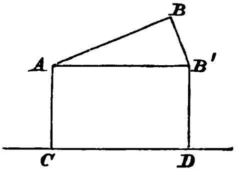
Again, $f(xy) = 0$ represents a locus or curve in the plane of xy; but, referred to three axes, it represents what is called a *cylindrical surface;* that is, a surface made by straight lines parallel to the axis of z drawn through every point in the curve or locus representing $f(xy) = 0$. For example, if $f(xy) = 0$ represents a circle on the plane of xy, it represents, when referred to three axes, a cylinder parallel to the axis of z cutting the plane of xy in that circle. If $f(xy) = 0$ represents an ellipse on the plane of xy, it represents an elliptical cylinder or tube similarly situated. If $f(xy) = 0$ is the equation to a hyperbola, then it represents a surface parallel to the axis of z, such that the section of it by any plane parallel to the coordinate plane of xy will be the hyperbola represented by the equation $f(xy) = 0$.

$x =$ a constant, manifestly represents a plane parallel to the plane of yz. Because, for any point in that plane the distance from the plane of yz (*i.e.*, the x of the point) is the same. Consequently $f(x) = 0$ represents a plane or a series of planes parallel to the plane of yz, according as the equation has one or more roots.

PROJECTION.

145. When from the extremities of one straight line two parallel lines are drawn to meet another straight line, the first straight line is said to be *projected* on the second; and the intercept made by the two parallel straight lines on the second straight line is called the projection of the first straight line.

When the two parallel straight lines are at right angles to the second straight line, it is called *orthogonal* projection. In that case the length of the projection is equal to the length of the original line multiplied by the cosine of the angle between the two straight lines. This is evident if the straight lines are in the same plane. If not, then from A, the extremity of the first line AB, draw AB' parallel to the second straight line CD. Then, letting fall BB' perpendicular to AB', we have $AB' = AB \cos BAB'$, and the angle BAB' is the inclination of the straight lines AB and CD. Let a plane be drawn through BB' perpendicular to CD, and another parallel plane through A; the intercept on CD will be the projection of AB upon it. This intercept is manifestly equal to AB', that is, equal to $AB \cos \theta$, where θ is the inclination of the given straight lines.



In projection, the *length* of a given straight line AB is projected on another straight line CD whose *direction* is given.

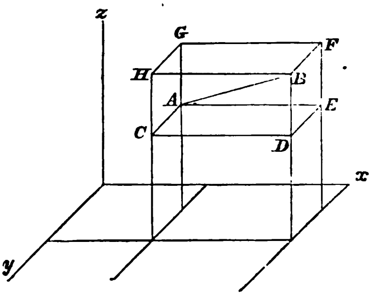
N.B.—Only rectangular coordinates and orthogonal projection will be used in the following pages.

If r be the distance of any point from the origin, and x, y, and z its coordinates, then x, y, and z are the projections of r on their respective axes. Thus, if α, β, and γ are the angles that the straight line r makes with the axes of x, y, and z respectively, then

$$r \cos \alpha = x, \quad r \cos \beta = y, \quad \text{and} \quad r \cos \gamma = z.$$

146. *To find the distance between two points whose coordinates are given.*

Let A be a point whose coordinates are x_1, y_1, z_1; and let B be another point whose coordinates are x_2, y_2, z_2; draw these coordinates, and complete the parallelepiped $ACDEFGHB$, of which they form the edges, and of which AB will be the diagonal. Then



$$AB^2=AD^2+BD^2$$
$$=AC^2+CD^2+BD^2.$$

But AC is manifestly y_2-y_1, and CD is x_2-x_1, and BD is z_2-z_1. Consequently the distance between the points A and B is equal to

$$\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]}.$$

If one of the points is the origin, then its coordinates are nothing. Thus, if r be the distance from the origin of a point whose coordinates are x_1, y_1, z_1,

$$r^2 = x_1^2 + y_1^2 + z_1^2.$$

147. *Relation between the cosines of the angles made by a straight line with the axes.*

If the straight line does not pass through the origin, draw a straight line parallel to it through the origin which will make the same angles with the axes. Take any point on the second straight line, and let its length from the origin be r, and its coordinates x, y, and z. By the foregoing article $r^2=x^2+y^2+z^2$; and by the article before that $x^2+y^2+z^2 = r^2\cos^2\alpha+r^2\cos^2\beta+r^2\cos^2\gamma$, where α, β, and γ are the angles made by the straight line with the axes.

Therefore $\qquad r^2 = r^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$;

therefore $\qquad \cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1.$

This is a very important and useful property.

The angles made by a straight line with the axes are the complements of the angles made with the coordinate planes; the axes being at right angles to the coordinate planes. For example, α is the angle the straight line makes with the axes of x, or the complement of the angle it makes with the plane of yz.

148. *To find the angle between two straight lines in terms of the angles the straight lines make with the axes.*

Through the origin draw two straight lines parallel to the given ones; and the angle between them will be equal to the required angle. Take any point on one of the straight lines through the origin, and let its coordinates be x_1, y_1, z_1, and its distance from the origin r_1: take any point on the other straight line through the origin, and let its coordinates be x_2, y_2, z_2, and its distance from the origin r_2: let the straight line joining these two points be called δ; and the angle between the straight lines be θ. Then, obviously,

$$\delta^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta,$$

and

$$\delta^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.$$

Therefore

$$r_1^2 + r_2^2 - 2r_1r_2\cos\theta$$

$$= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2(x_1x_2+y_1y_2+z_1z_2).$$

But $r_1^2 = x_1^2 + y_1^2 + z_1^2$, and $r_2^2 = x_2^2 + y_2^2 + z_2^2$;

therefore

$$r_1r_2\cos\theta = x_1x_2+y_1y_2+z_1z_2.$$

Substituting for x_1, y_1, z_1 and x_2, y_2, z_2 their values in terms of r_1 and r_2 and $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ (these being the angles the straight lines make with the axes), we have

$$r_1r_2\cos\theta = r_1r_2(\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2),$$

or

$$\cos\theta = \cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2.$$

149. We may now proceed to examine how a straight line is represented with coordinates of three dimensions. Let a, b, c be the known coordinates of a point through which the line passes, and let x, y, z be the coordinates of any point on the line, and let the distance between the two points be r. The projection of that length r on the axis of x is $r\cos\alpha$, and on the axis of y is $r\cos\beta$, and on the

axis of z is $r \cos \gamma$ (α, β, and γ being the angles the straight line makes with the axes). But $x-a$ is also the projection of r on the axis of x, as it is the intercept on the axis of x by two planes perpendicular to and passing through the extremities of r, which are the points (a, b, c) and (x, y, z). Therefore $r \cos \alpha = x-a$; similarly $r \cos \beta = y-b$, and $r \cos \gamma = z-c$. From these we get

$$r = \frac{x-a}{\cos \alpha} = \frac{y-b}{\cos \beta} = \frac{z-c}{\cos \gamma}.$$

Since the point (x, y, z) is indefinite, the length r is indefinite, and the relation becomes

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N},$$

where L, M, and N are proportional to the cosines of the angles the straight line makes with the axes, and a, b, c are the coordinates of a point through which it passes. These three equations are in reality only two, for $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are connected by the property proved in the previous article, that the sum of their squares is equal to 1; and therefore the L, M, and N are connected by this property that $L^2+M^2+N^2 =$ a constant. If any two of these are given the third is determined.

N.B.—Cos α, $\cos \beta$, and $\cos \gamma$ are called the *direction cosines* of the straight line which makes the angles α, β, and γ with the axes.

The three equations

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}$$

can be reduced to two, viz. $x = Az+p$;

where $\quad A = \frac{L}{N}$, and $p = a-\frac{L}{N}c$:

and $\quad y = Bz+q$;

where $\quad B = \frac{M}{N}$, and $q = b-\frac{M}{N}c$.

It will be shown afterwards that a simple equation represents a plane when referred to axes in three dimensions;

and the straight line represented by these two equations is the intersection of the planes they each represent. The first equation represents a plane parallel to the axis of y, the second a plane parallel to the axis of x.

As, in plane coordinate geometry, a point can only be represented as the intersection of two straight lines, that is, by two equations; so, in coordinate geometry of three dimensions, a straight line must be represented as the intersection of two planes.

The equations to a straight line passing through the origin are $\frac{x}{L} = \frac{y}{M} = \frac{z}{N}$; for a, b, c are each equal to nothing; the origin being a point through which the straight line passes.

If the straight line is parallel to the plane of yz it is at right angles to the axis of x, therefore $\alpha = \frac{\pi}{2}$, and $\cos\alpha = 0$: from this $\frac{x-a}{0} = r$, we have $x-a = 0$. Therefore the equations to a straight line parallel to the plane of yz are $x = a$, $\frac{y-b}{M} = \frac{z-c}{N}$; M and N being as before proportional to the cosines of the angles made by the straight line with the axes of y and z.

Similarly, if the straight line is parallel to the axis of z, the equations become $x = a$, $y = b$: for the expression $\frac{z-c}{N} = \frac{y-b}{0}$ becomes indeterminate, and very obviously $x = a$ represents a plane parallel to the plane of yz, and $y = b$ a plane parallel to the plane of xz, and the straight line represented by the two equations is that formed by the intersection of the two planes.

150. *To find the equations of a straight line passing through two points.*

Let x_1, y_1, z_1 be the coordinates of one point, and x_2, y_2, z_2 the coordinates of the other. Then, since the straight line passes through the first point, its equations must be of the form $\frac{x-x_1}{L} = \frac{y-y_1}{M} = \frac{z-z_1}{N}$; L, M, and N being indeterminate. Again, since the straight line passes through the second point, its equations must be satisfied by the coordinates of it; therefore $\frac{x_2-x_1}{L} = \frac{y_2-y_1}{M} = \frac{z_2-z_1}{N}$.

Dividing these equations (which are all equal to the same indeterminate), we have

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

These are the equations to the straight line passing through the given points, and they are manifestly satisfied by the coordinates of either of them.

151. *To find the values of the direction cosines in terms of L, M, and N.*

$$L = k \cos \alpha\,; \; M = k \cos \beta\,; \; N = k \cos \gamma.$$

Therefore

$$L^2+M^2+N^2 = k^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = k^2.$$

Since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$

therefore $L = \sqrt{L^2+M^2+N^2} \cos \alpha,$

or $\cos \alpha = \frac{L}{\sqrt{(L^2+M^2+N^2)}}.$

Similarly $\cos \beta = \frac{M}{\sqrt{(L^2+M^2+N^2)}};$

and $\cos \gamma = \frac{N}{\sqrt{(L^2+M^2+N^2)}}.$

152. *To find the condition that two straight lines should be parallel or perpendicular.*

Let the equations of the straight lines be

$$\frac{x-a}{L} = \frac{y-b}{M} = \frac{z-c}{N}, \text{ and } \frac{x-a'}{L'} = \frac{y-b'}{M'} = \frac{z-c'}{N'}.$$

If the lines are parallel the direction cosines are the same; consequently L, M, and N must be proportional to L', M', and N'; or $\frac{L}{L'} = \frac{M}{M'} = \frac{N}{N'}$.

If θ be the angle between the two straight lines, we have $\cos\theta = (\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$,

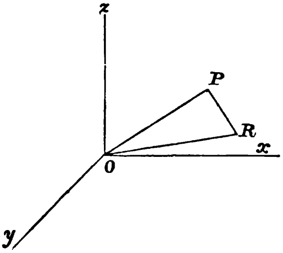
or
$$\cos\theta = \frac{LL' + MM' + NN'}{\sqrt{(L^2+M^2+N^2)(L'^2+M'^2+N'^2)}}.$$

If the straight lines are perpendicular, $\cos\theta = 0$; therefore
$$LL' + MM' + NN' = 0.$$

This is the condition that the two straight lines should be at right angles.

EQUATION TO A PLANE.

153. The equation can be easily found in terms of the length of the perpendicular from the origin on the plane, which is termed the *normal* of the plane, and the angles which this perpendicular makes with the axes.



Let these angles be called α, β, and γ; and the length of the normal p. In the figure let P be the foot of the perpendicular OP from the origin on the plane; then the coordinates of P will be (Art. 145) $x = p\cos\alpha$, $y = p\cos\beta$, $z = p\cos\gamma$. Let the coordinates of any other point R in the plane be x, y, z; then the length of RP, the line joining these two points, is (Art. 139)

$$(x - p\cos\alpha)^2 + (y - p\cos\beta)^2 + (z - p\cos\gamma)^2.$$

But $OR^2 = x^2 + y^2 + z^2$; and also $OR^2 = OP^2 + RP^2$, since OP is perpendicular to every straight line in the plane.

Consequently, as $OR^2 = OP^2 + RP^2$,

we have

$$x^2+y^2+z^2 = p^2+(x-p\cos\alpha)^2 \\ +(y-p\cos\beta)^2+(z-p\cos\gamma)^2,$$

or $$x^2+y^2+z^2 = p^2+x^2+y^2+z^2-2p(x\cos\alpha+y\cos\beta+z\cos\gamma) \\ +p^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

But this coefficient of p^2 is equal to unity, therefore the equation becomes

$$2p(x\cos\alpha+y\cos\beta+z\cos\gamma) = 2p^2,$$

or $$x\cos\alpha+y\cos\beta+z\cos\gamma = p.$$

This is the simplest form of the equation to the plane. In general $Ax+By+Cz = D$ represents a plane, A, B, and C being proportional to the cosines of the angles the normal makes with the axes, and D to the length of the normal.

The equations to the normal obviously are (Art. 149)

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma}, \text{ or } \frac{x}{A} = \frac{y}{B} = \frac{z}{C},$$

where A, B, and C are proportional to $\cos\alpha$, $\cos\beta$, and $\cos\gamma$.

The angle between two planes is the same as the angle between their normals, consequently (Art. 152) the condition that two planes

$$Ax+By+Cz = D \text{ and } A'x+B'y+C'z = D'$$

should be perpendicular, is

$$AA'+BB'+CC' = 0.$$

The condition that two planes should be parallel is evidently that their equations should differ only in the indeterminate coefficient, as they have the same normal, but of different length.

The angles which a plane makes with the coordinate planes are the same as the angles which the normal makes with the axes.

For example, the normal is perpendicular to the plane, and the axis of x is perpendicular to the coordinate plane of yz. Therefore the angle the plane makes with the coordinate plane of yz is the angle the normal makes with the axis of x.

154. *Equation of a plane in terms of the intercepts it makes on the axes.*

This is a very useful form of the equation to a plane, analogous to that of a straight line in Art. 12; and can be found thus.

The intercept made by the plane on the axis of x is obtained by making $y=0$ and $z=0$; from which we have $Ax=D$; or the intercept on the axis of x is $\frac{D}{A}$. Similarly the intercept on the axis of y is $\frac{D}{B}$; and on the axis of z is $\frac{D}{C}$. Let these intercepts be respectively a, b, and c. Dividing the equation of the plane by D, we have

$$\frac{A}{D}x+\frac{B}{D}y+\frac{C}{D}z=1$$

or, since $\frac{A}{D}=\frac{1}{a}$, &c.,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

From this form of the equation it is easy to find the equation of a plane passing through three points whose coordinates are given. For, putting in the successive values of x, y, and z, three equations are given to find a, b, and c.

EQUATION OF A SPHERE.

155. If a sphere have its centre at the origin, and its radius equal to r, then for any point on its surface

$$x^2+y^2+z^2=r^2.$$

This is the equation to the sphere; and this will not be changed, no matter how the axes are drawn through the centre as origin provided that they are rectangular.

If the origin be not at the centre, then let three parallel axes be drawn through the centre, and the equation of the sphere referred to them is

$$x^2+y^2+z^2 = r^2.$$

To transform this to the given axes, we must for x substitute $x+a$; for y substitute $y+b$; and for z substitute $z+c$; where a, b, and c are the coordinates of the centre referred to the given axes.

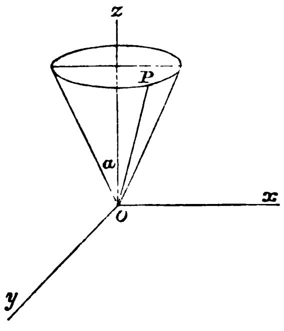
The equation becomes

$$(x+a)^2+(y+b)^2+(z+c)^2 = r^2,$$

or $$x^2+y^2+z^2+Ax+By+Cz+D = 0.$$

This is the general form of the equation to the sphere (*cf.* general form of the equation to the circle in Article 40).

EQUATION TO A RIGHT CONE.



156. Let the generating angle of the cone be α; and let it be described round the axis of z as its axis of revolution, and let the origin be at the vertex. If P be any point on the surface of the cone, its coordinates being x, y, and z, a plane through P parallel to the plane of xy will make an intercept on the axis of z equal to the z of the point P. Let r be the distance of P from the origin, then

$$r\cos\alpha = z, \quad\text{or}\quad r = z\sec\alpha.$$

Therefore $$r^2 = z^2\sec^2\alpha,$$

or $$x^2+y^2+z^2 = z^2(1+\tan^2\alpha),$$

or $$x^2+y^2 = z^2\tan^2\alpha.$$

Therefore $x^2+y^2=k^2z^2$ represents a right cone whose vertex is at the origin, whose axis is the axis of z, and where k is the tangent of the generating angle.

Examples.

1. Find the angle between the straight lines whose direction cosines are proportional to 1, 2, 3, and to 2, 3, 6.

[*Ans.* $\cos^{-1}\dfrac{4}{\sqrt{17}}$.

2. Find the angle between the straight lines whose direction cosines are proportional to 1, 2, 3, and to 5, −4, 1.

[*Ans.* A right angle.

3. Find the equations of the straight line passing through the point (1, 2, 3), and whose direction cosines are proportional to $\sqrt{3}$, 1, and $2\sqrt{3}$.

[*Ans.* $\dfrac{x-1}{\sqrt{3}}=y-2=\dfrac{z-3}{2\sqrt{3}}$.

4. Find the equations to the sides of the triangle the coordinates of whose angles are (1, 2, 3), (3, 2, 1), and (2, 3, 1).

Ans. $\dfrac{x-1}{2}=\dfrac{3-z}{2},\ y=2;$

$3-x=y-2,\ z=1;$

$x-1=y-2=\dfrac{3-z}{2}.$

5. Find the angles of the triangle. [*Ans.* 60°, 90°, 30°.

6. Also the equation of the plane of it, and the length of the normal. [*Ans.* $x+y+z=6$; normal $=2\sqrt{3}$.

7. Find the equation of the plane perpendicular to the plane $x+2y+3z=8$, and passing through the point (1, 2, 3). [*Ans.* $6x+3y+2z=18$.

8. In the equation to the cone, find what takes place when the generating angle becomes 0, or 90°.

9. A point moves so that the ratio of its distances from two fixed points is constant; find its locus.

[*Ans.* A sphere.

10. A point moves so that the straight lines joining it to two fixed points are at right angles, find its locus. Let the points be (3, 7, 4) and (5, 3, 6); find the equation.

[*Ans.* A sphere with the straight line joining the given points as diameter.

$$x^2+y^2+z^2-8x-10y-10z+66=0.$$

APPENDICES.

APPENDIX I.

A COLLECTION OF USEFUL PROBLEMS AND THEOREMS ON THE STRAIGHT LINE AND CIRCLE, WITH HINTS FOR THEIR SOLUTION.

I. FIND a point in a straight line such that the straight lines joining it to two given points shall make equal angles with the given straight line; and shew that the sum of these two straight lines is less than the sum of the straight lines from any other point in the given straight line to the two given points.

From one of the given points let fall a perpendicular on the given straight line, and produce it on the other side till equal to itself; join its extremities with the other given point: the point where the joining line cuts the given straight line is the point required.

II. Draw through a given point between two straight lines which are not parallel a straight line which will be bisected in that point.

Draw from the given point two straight lines parallel to the given straight lines; draw the diagonal of the parallelogram so formed: a straight line parallel to it through the given point is the required straight line.

III. Prove that the three straight lines from the angles of a triangle to the middle points of the sides meet in a point, which cuts one-third of each above the sides.

If two bisectors of the sides are drawn, it may be shewn without difficulty that each cuts one-third of the other from the sides: from this it follows at once that they meet in a point.

IV. The areas of the three triangles, with this point as common vertex and the sides of the triangle as bases, are equal.

For each is obviously one-third of the whole triangle.

V. If from any point in the plane of a parallelogram straight lines be drawn to the extremities of two sides and a diagonal that are conterminous, of the triangles so formed with this point as vertex that whose base is the diagonal will be equal to the sum or difference of the two whose bases are the sides, as the point is outside the parallelogram, or inside it, or between these or the other two sides produced.

Take the straight line joining the given point and the angle of the parallelogram as the common base of the three triangles, and compare the lengths of the perpendiculars of the triangles on this common base.

VI. The difference of the squares of the sides of a triangle is equal to twice the rectangle between the base and the segment from the middle point of the base to the foot of the perpendicular on it.

For the difference of the squares of the sides is equal to the difference of the squares of the segments of the base made by the perpendicular, and this is equal to the rectangle under the sum and difference of the segments.

VII. The sum of the squares of the sides of a triangle is equal to twice the square of half the base together with twice the square of the 'straight line joining the middle point of the base with the opposite angle.

This follows at once from Euclid, Book II., Props. 12 and 13.

VIII. The sum of the squares of the sides of a quadrilateral figure is equal to the sum of the squares of the diagonals together with four times the square of the straight line joining the middle points of the diagonals.

Join the middle point of one of the diagonals with the opposite angles, and apply the preceding proposition to the two triangles into which this diagonal divides the quadrilateral.

IX. The square of any straight line drawn from the vertex to the base of an isosceles triangle is less or greater than the square of the side of the triangle by the rectangle under the segments of the base, as the straight line cuts the base internally or externally.

For the difference between the squares of the line and the side is equal to the difference of the squares of the segments on the base from the foot of the perpendicular, and this is equal to the rectangle under the sum and difference of those segments.

X. The sum of the perpendiculars from any point in an equilateral triangle is constant. If the point be outside the triangle, the difference between one perpendicular and the sum of the other two is the same constant.

For the sum of the triangles with this point as vertex and the sides as bases, or the difference between one of these triangles and the sum of the other two, is the original triangle.

XI. Given the base, vertical angle, and (1) area, or (2) the difference, or (3) the sum of squares of sides, construct the triangle.

A segment of a circle described on the given base, and containing an angle equal to the given vertical angle, is one locus of the vertex; if the area is given, a straight line parallel to the base is another locus of it; if the difference of the squares of the sides is given, then let the base be cut so that the difference of the squares of the segments be equal to the given difference, and a perpendicular at the point of section is the other locus; if the sum of the squares of the sides be given, a circle with centre at the middle point of the base, and radius of such length that twice its square together with twice the square of half the base is equal to the given sum, is the other locus—compare Prop. VII. —and the intersection of the two loci determines the vertex.

XII. Given base, vertical angle, and (1) sum, or (2) difference of sides, construct the triangle.

On the given base describe a segment of a circle containing an angle equal to half the given vertical angle; from one extremity of the base inflect on the segment a straight line equal to the given sum; or if the difference is given, on the base describe a segment containing an angle equal to the supplement of half the supplement of the given vertical angle, inflect on this a straight line equal to the given difference; the rest is easily done.

N.B.—The student should observe when these conditions make the triangle a maximum, a minimum, or impossible.

XIII. Given the vertical angle of a triangle and the sum of the two sides containing it, and that the perpendicular or the bisector of the angle or that of the base passes through a given point, &c., construct the triangle.

Take on the sides of the angle lengths each equal to half the given sum, describe a circle touching the sides in these two points, the base must touch this circle, &c.

XIV. What is the condition that a quadrilateral can have a circle inscribed in it?

The sums of the opposite sides must be equal.

XV. The perpendiculars from the angles of a triangle on the sides meet in a point.

Draw two of them, and join their intersection with the third angle, and produce it to meet the third side; a semicircle on this side will pass through the feet of the two perpendiculars; also a circle will pass through the feet of these two perpendiculars, their intersection, and the third angle: by these two circles the proposition may be easily proved.

XVI. To describe a circle that shall pass through two given points, and touch (1) a given straight line, or (2) a given circle.

(1) Join the points, and produce the joining line to meet the given line, and from the point where they intersect take a length on the given line such that its square shall be equal to the rectangle under the distances of the given points from the point of intersection: the circle through the given points and the point so found is that required (Euc., Book III., 37).

(2) Describe any circle through the given points to cut the given circle, produce the common chord of these circles to meet the straight line joining the given points, and from the point of intersection draw a tangent to the given circle: a circle through the given points and the point of contact of this tangent is that required.

XVII. To describe a circle that shall touch a given circle and a given straight line in a given point.

Draw a radius of the circle perpendicular to the given straight line, and join the extremity of this radius with the given point: where this joining line meets the circumference again is the point where the required circle will touch the given circle.

XVIII. To describe a circle that shall pass through a given point and touch two given straight lines.

Join the given point with the intersection of the given straight lines, describe any circle touching the two given straight lines, join its point of contact with the point where it cuts the straight line joining the given point and the intersection of the given straight lines: a straight line from the given parallel to the line so found, to meet one of the given lines, will give the point in which the required circle will touch that straight line.

XIX. To draw a common tangent to two given circles.

Join the centres, and on the joining line describe a semicircle; inflect in this, from the centre of the greater circle, a straight line equal to the difference or sum of the radii; join this with the centre of the other circle: this joining line will be parallel to the required common tangent.

N.B.—The last four problems have each two solutions.

XX. Find the locus of points from which equal tangents can be drawn to two given circles.

It can be easily seen that this is equivalent to finding the locus of points; the difference of the squares of the straight lines joining them with the centres is equal to the difference of the squares of the radii. This is called the radical axis of the circles.

XXI. If from any point on the arc of a circle perpendiculars be let fall on the bounding radii, the distance between their feet is constant.

For it subtends a fixed angle (the angle between the bounding radii) in a constant circle whose diameter is the radius of the arc.

XXII. If a circle roll inside one of double the radius, any point on it will describe a diameter of the fixed circle.

Take any position of the rolling circle, and through any point on it draw a diameter of the fixed circle, the arcs of both circles from the points where this diameter meets them to the point of contact may be easily shown to be equal; and from this the proposition can be proved.

Cor.—A ladder is gradually raised against a wall, the locus of its middle point is easily seen to be a quadrant drawn from the foot of the wall as centre, with radius equal to half the length of the ladder.

XXIII. Given two points and a straight line, find the point where the given points subtend the greatest angle.

Describe a circle through the two given points to touch the given straight line (Prop. XVI.): the point of contact is that required.

XXIV. Find the locus of points dividing in a given ratio the straight lines drawn from a given point to a given straight line.

XXV. Find the locus of points dividing in a given ratio (1) the chords drawn from a point in the circumference of a circle; (2) the straight lines from any given point to the circumference of a circle.

(2) Draw from the given point a diameter of the circle; cut the two segments of it from the point to the circumference in the given ratio: the circle on the portion between these points of section as diameter is the required locus. (1) Is obvious.

XXVI. If the vertical angle of a triangle be bisected, the rectangle under the sides is equal to the rectangle under the segments of the base together with the square of the bisector of the vertical angle.

Circumscribe the triangle, and produce the bisector to meet the circumference of the circle, and join that point with either extremity of the base. Then there are two similar triangles,—one formed by one side, the bisector, and segment of base; the other formed by the bisector produced, the other side, and the joining line: from this the proposition follows.

XXVII. The rectangle under the sides of a triangle is equal to the rectangle under the perpendicular on the third side, and the diameter of the circumscribing circle.

Draw the diameter from the angle from which the perpendicular is drawn; join the other end of the diameter with either extremity of the base. Then there are two similar triangles formed, as in previous proposition.

XXVIII. The rectangle under the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles under the opposite sides.

Draw a straight line at an angle of the quadrilateral making with one side an angle equal to that made by the conterminous diagonal and the other conterminous side. From two sets of similar triangles thus formed the proposition can be proved.

XXIX. If from any point on the circumference of a circle perpendiculars be let fall on the sides of a quadrilateral inscribed in it, the products of the perpendiculars on the opposite sides are equal.

Join the feet of the perpendiculars on the adjacent sides: by this two similar triangles are formed, from which the proposition follows.

N.B.—From this a very easy method of finding the equation to the circle described on the straight line joining two given points as diameter can be deduced. Let the given points be (x_1, y_1), and (x_2, y_2), and let (x, y) be any point on the circle; draw through the given points straight lines parallel to the axes, these will form a rectangle inscribed in the circle. By this proposition, we have

$$(x-x_1)(x-x_2) = (y_1-y)(y-y_2),$$

which is the equation to the circle. See Chapter V.

XXX. (1) Inscribe a square in a triangle; and (2) show that its side is half the harmonic mean between the base and perpendicular of the triangle.

(1) At the extremity of the base draw a perpendicular equal to it, and join its extremity with the vertex; where this joining line cuts the base, draw a perpendicular to meet the side, that will be a side of the square. (2) The ratio of the base of the triangle to the perpendicular is manifestly equal to the ratio of the side of the square to the perpendicular *minus* the side of the square.

XXXI. Given base and ratio of the sides, find the locus of the vertex of the triangle.

Since the bisectors of the internal and external vertical angle cut the base and the base produced in the given ratio, and include a right angle between them, the vertex must lie on a semicircle on the segment between the points of section so found. (Euc., Bk. VI., Prop. 3.)

XXXII. If the vertical angle of a triangle be bisected internally and externally, the segments of the base measured either from the point of external section, or from the extremity of the base remote from that point, will be in harmonical progression.

This follows also from Euc., Bk. VI., Prop. 3 and Cor.: taking this definition of harmonic progression, the first is to the third as the difference between the first and second is to the difference between the second and third.

XXXIII. If a straight line be cut harmonically, and the angle contained by the straight line drawn from any point to an extremity and one point in the line be bisected by the straight line drawn to the other point in the line, then this straight line will be at right angles to the straight line drawn to the other extremity of the line.

XXXIV. If a straight line drawn across four straight lines that meet in a point be cut by them harmonically, then every other straight line drawn across them will be cut harmonically.

Let O be the point in which the four lines meet, and A, B, C, D the points in which the given straight line is cut harmonically by them, and A', B', C', D' the points in which any other straight line is cut by them; then (Euc., Bk. VI., Prop. 23) the triangle

$$AOB : A'OB' = AO \,.\, OB : A'O \,.\, OB';$$

and

$$COD : C'OD' = CO \,.\, OD : C'O \,.\, OD';$$

$$\therefore \quad \frac{AOB \times COD}{A'OB' \times C'OD'} = \frac{AO \,.\, BO \,.\, CO \,.\, DO}{A'O \,.\, B'O \,.\, C'O \,.\, D'O}.$$

Similarly

$$\frac{BOC \times AOD}{B'OC' \times A'OD'} = \frac{AO \,.\, BO \,.\, CO \,.\, DO}{A'O \,.\, B'O \,.\, C'O \,.\, D'O};$$

$$\therefore \quad \frac{AOB \,.\, COD}{A'OB' \,.\, C'OD'} = \frac{BOC \,.\, AOD}{B'OC' \,.\, A'OD'};$$

and (Euc., Bk. VI., Prop. 1)

$$\frac{AB \,.\, CD}{A'B' \,.\, C'D'} = \frac{BC \,.\, AD}{B'C' \,.\, A'D'};$$

or the proportion in which the line $A'B'C'D'$ is cut, is the same as that in which $ABCD$ is cut.

XXXV. If a straight line drawn parallel to one of four straight lines that meet in a point has intercept between the first and third bisected by the second, all the straight lines meeting those four will be cut harmonically.

Draw any straight line through the point where the line parallel to the fourth is bisected by the second; then (Euc., Bk. VI., Prop. 2), the segments of that line will be found to bear to each other the ratio of the intercept made on the fourth to the two equal intercepts of the line parallel to the fourth; *i.e.*, the whole line to the first segment as the third segment to the middle or second segment.

XXXVI. If from any point in a straight line tangents be drawn to a given circle, the chords of contact will pass through a fixed point.

XXXVII. If through any point in a circle chords be drawn, the tangents at the extremities of each of them will intersect in a straight line.

For these two examples see Art. 49, Chapter V.

APPENDIX II.

A COURSE OF ELEMENTARY PROPOSITIONS ON SOLID GEOMETRY, WHICH MAY BE SUBSTITUTED FOR THOSE IN THE ELEVENTH BOOK OF EUCLID.

[*The definitions of that Book are assumed, and where the propositions admit of the same proof it will be indicated.*

I. Only one plane can be drawn through—

(*a*) three points not in a straight line;
(*b*) two intersecting straight lines;
(*c*) two parallel straight lines.

This is easily seen by supposing a plane to pass through one pair of points, on one straight line, and revolving to pass through or intersect the other.

II. Two planes intersect in a straight line.

Euclid, Book XI., Prop. 3.

III. If a straight line is perpendicular to straight lines at their intersection, it is perpendicular to any straight line in their plane.

This may be much more easily proved than in Euclid, Book XI., Prop. 4, by taking equal lengths on the perpendicular above and below the intersection, and joining its extremities with points in the given lines and where the line joining the given point cuts any line in the plane through the intersection of the given lines; and then using the same method.

IV. There cannot be two perpendiculars to the same plane from the same point.

Euclid, Book XI., Prop. 13. From a point above the plane it evidently follows from Euclid, Book I., Prop. 16.

V. If two straight lines are perpendicular to the same plane, they are parallel to each other.

Euclid, Book XI., Prop. 6.

VI. Through a given point only one plane can be drawn parallel to two given straight lines.

For if through the point two straight lines be drawn parallel to the given lines, only one plane can be drawn through these lines. See above, Prop. 1. (*b*).

VII. Planes to which the same straight line is perpendicular are parallel.

Euclid, Book XI., Prop. 14.

VIII. If two parallel planes are intersected by a third plane, then common sections are parallel straight lines.

Euclid, Book XI., Prop. 16.

IX. If two straight lines be cut by parallel planes, they are cut in the same ratio.

Euclid, Book XI., Prop. 17. Definition.—The projection of a point on a plane is the foot of the perpendicular let fall from the point on the plane. Hence the projection of a straight line is the locus of the feet of the perpendiculars let fall from any point in that straight line on the plane.

X. The projection of a straight line is a straight line.

XI. The angle which a straight line makes with its projection on a plane is less than that it makes with any other straight line in the plane.

Let the line and its projection be produced to meet; take any point on the line, and any point on its projection, and a point equally distant on any straight line in the plane through the intersection of the line and projection. Joining these two points with the point on the straight line, it may easily be shewn (Euclid, Book I., Prop. 25) that the angle between the line and its projection is the least.

XII. Given two straight lines in space not in the same plane, draw the common perpendicular, and prove that it is the shortest distance between them.

Through the first straight line draw a plane parallel to the second; from any point in the second let fall a perpendicular on the plane; through its foot draw a straight line parallel to the second to meet the first: from this the common perpendicular is drawn.

XIII. If a solid angle be contained by three plane angles meeting in a point, any two of them are greater than the third

Euclid, Book XI., Prop. 20.

XIV. Every solid angle is contained by plane angles which are together less than four right angles.

Euclid, Book XI. Prop., 21.

XV. There are only five regular solids.

For the only regular rectilineal figures whose angles can be put together to make a solid angle are—(1) three equilateral triangles; (2) three squares; (3) four equilateral triangles; (4) three regular pentagons; (5) five equilateral triangles. (There cannot be more, as the sum of the plane angles must be together less than four right angles.) These form respectively the solid angles or vertices of (1) a tetrahedron (four sides); (2) a cube (six sides); (3) an octahedron (eight sides); (4) a dodecahedron (twelve sides); (5) an icosahedron (twenty sides.)

XVI. The square of the diagonal of a rectangular parallelepiped is equal to the sum of the squares of its three edges.

XVII. The volume of a pyramid is one-third of the product of its base and altitude.

Taking triangular pyramids (into which every pyramid can be divided), it can be shewn that those with equal bases and altitudes are equal; for they may be divided into an infinitely great number of equal prisms by planes parallel to their bases.

These triangular prisms may be divided into three triangular pyramids of equal bases and altitudes, and therefore of equal volume. Thus, let ABC be the base of the prism, and D, E, F the top, the three pyramids are (making the vertex the last letter) $ABCE$, $DBFC$, *ABCD*. And a triangular pyramid can be made into a prism of the same base and altitude of three times the volume. As the volume of the prism is equal to the product of the base and altitude, the volume of the pyramid is one-third of that.

The student should write out full demonstrations of these propositions from the hints given.

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